THE TUTORIAL TRIGONOMETRY

BRIGGS AND BRYAN

THE TUTORIAL TRIGONOMETRY

THE

TUTORIAL TRIGONOMETRY

BY

WILLIAM BRIGGS, LL.D., M.A., B.Sc.

AND

G. H. BRYAN, Sc.D., F.R.S.

LATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY COLLEGE OF NORTH WALES

Authors of "The Right Line and Circle," etc.

UNIVERSITY TUTORIAL PRESS LTD

Clifton House, Euston Road, London, N.W. 1

Third Edition 1928
Reprinted 1933, 1935, 1940, 1942, 1945, 1947, 1950, 1954, 1957

PRINTED IN GREAT BRITAIN BY UNIVERSITY TUTORIAL PRESS LTD, FOXTON
NEAR CAMBRIDGE

PREFACE.

THE subject matter of the present book naturally falls under three different headings. The first ten chapters deal mainly with what has been designated at Cambridge by the not over appropriate title of "Trigonometry of One Angle," the next four chapters treat the trigonometry of two or more angles, and the remainder of the book is devoted to logarithms and the trigonometry of triangles.

It has become somewhat fashionable of recent years to defer all considerations of algebraic sign till a number of trigonometric identities have been dealt with at considerable length for acute angles. Experience, however, shows that when students have once become thoroughly familiar with the restricted and unsatisfactory definitions, it is usually too late for them to discard these in favour of the general definitions. For this reason the trigonometric functions are defined generally at the earliest possible stage in the work.

It must not be forgotten, too, that in order to acquire a sound knowledge of trigonometry, a thorough grasp of the nature and general properties of trigonometric functions is just as essential as facility in manipulating trigonometric expressions. In the preparation of the earlier chapters, the former requirement has been kept prominently in view, while for the latter purpose the very large number of examples for exercise should furnish the reader with ample material on which to gain proficiency.

The "Illustrative Exercises" given in the text call for some explanation. There can be no better way of becoming familiar with the bookwork of a subject than by reproducing it with some slight modifications of form or notation (such as are very usually introduced in examination questions), and it is the object of these illustrative exercises to supply such suggestions as will enable readers to do this for themselves.

For example, where, in the text, sexagesimal measure is used, the reader is asked to reproduce the proof, using circular measure, and so on. In other cases, the exercises consist of perfectly simple questions which the reader should ask himself before proceeding further. A few bookwork questions have been introduced among the examples themselves.

The comparative importance of the subject-matter is indicated by the type used, and fundamental propositions which the reader should be able to reproduce have the numbers of the paragraphs as well as their headings in dark type (thus **27**). Where articles or examples are denoted by an asterisk it is usually implied that they can be omitted on first reading: but in Chapter I. the "starred" section and examples refer to the obsolete "centesimal measure" of angles which may be omitted or read and worked merely as a matter of interest.

Owing to the great importance attached to graphic methods we have given special attention to this feature in the chapter on "Trigonometric Functions of a Variable Angle," and by geometric constructions for the curves of the sine, tangent, etc. "The Use of Tables" has also received due prominence.

NOTE TO THE THIRD EDITION.

In this edition the chapter dealing with the graphic representation of the trigonometric functions has been placed earlier in the book—as Chapter V. instead of Chapter X.—so that the relations between the trigonometric functions of allied angles may be illustrated by references to the corresponding graphs: this has necessitated a few minor alterations in the text of this chapter. The treatment of infinity has also been amended.

CONTENTS.

N.B.—The italic numerals refer to the pages on which the examples begin in each chapter.

In the following table, only a few of the most important formulae are enumerated. These should be remembered on first reading, and the student is also advised to draw up a list of ALL *the formulae which are numbered consecutively throughout the book.*

THE TUTORIAL TRIGONOMETRY

CHAPTER I.

SEXAGESIMAL MEASUREMENT OF ANGLES.

1. **Trigonometry** is a word derived from the Greek, signifying measurement of triangles. This was the original object of Trigonometry, and is still one of its most important applications, but at present the subject includes all that branch of mathematics which deals with **angles.**

In Geometry, angles are generally measured by the number of right angles or fractions of a right angle that they contain. In Trigonometry, several other methods of measuring angles are adopted, of which the so-called Sexagesimal Measure is the most important. The name *sexagesimal* refers to the fact that each unit is *sixty* times the next smaller one.

2. **The units of Sexagesimal Measure** are the sub-divisions of a right angle, defined as follows :—

$$\left.\begin{array}{lll} \text{1 right angle} & = 90 \textbf{ degrees}, & \text{denoted by } \mathbf{90^\circ} \\ \text{1 degree or } 1^\circ & = 60 \textbf{ minutes}, & \text{denoted by } \mathbf{60'} \\ \text{1 minute or } 1' & = 60 \textbf{ seconds}, & \text{denoted by } \mathbf{60''} \end{array}\right\} \quad \ldots\ldots\textbf{(1)}$$

Although these units are thus derived from the right angle, the right angle itself is not a sexagesimal unit, the largest unit being the *degree.* Thus, in sexagesimal measure, 2 right angles $= 180^\circ$, and so on.

3. **To reduce angles** from right angles to degrees, or from degrees to minutes and seconds, and *vice versa*, we proceed by the ordinary rules of arithmetic.

Ex. 1. Express $57{\cdot}296^\circ$ in degrees, minutes, and seconds.

Reduce $\cdot 296$ degrees to minutes by multiplying by 60; we get $17{\cdot}76'$. Reduce $\cdot 76$ minutes to seconds by multiplying by 60; we get $45.6''$.

Thus we have $\qquad 57^\circ\, 17'\, 45{\cdot}6''$.

Ex. 2. Reduce $46^\circ\, 6'\, 9''$ to the decimal of a right angle.

Rule.—Divide seconds by 60, and prefix the minutes. Divide again by 60, and prefix degrees. Divide by 90.

$$\begin{array}{l} 60\,\underline{)\;9''} \\ \quad \cdot 15' \qquad 60\,\underline{)\;6{\cdot}15'} \\ \qquad\qquad\quad \cdot 1025^\circ \qquad 90\,\underline{)\;46{\cdot}1025^\circ} \\ \qquad\qquad\qquad\qquad\qquad\quad \cdot 51225 \text{ rt. angle. } \textit{Ans.} \end{array}$$

Ex. 3. Express in sexagesimal measure each of the angles (i) of a regular pentagon, (ii) of a regular polygon of seven sides.

(i) Let A denote the angle of a regular pentagon. Then, by Euclid I. 32, Cor.,

$5A + 4$ rt. angles = twice as many rt. angles as there are sides;

$$\therefore \quad 5A = (10 - 4) \text{ rt. angles} = 6 \text{ rt. angles} = 540^\circ;$$
$$\therefore \quad A = 108^\circ.$$

(ii) In like manner, if A be the angle of the heptagon, we have $\qquad 7A + 4 \times 90^\circ = 14 \times 90^\circ$, or $7A = 900^\circ$.

$$7\,\underline{)\;900^\circ}$$
$$\therefore \quad \text{required angle } A = 128^\circ\, 34'\, 17\tfrac{1}{7}''.$$

***4.** *Centesimal Measure.*—Another mode of measuring angles was proposed at the time of the French Revolution, when the decimal system of weights and measures was introduced. It never came into general use, and is now obsolete.†

A grade ($\frac{1}{100}$th of a right angle) was the unit of Centesimal Measure. It was subdivided into 100 (centesimal) minutes or primes, and each prime into 100 seconds.

---

* Articles marked with an asterisk may be omitted on a first reading.

† Questions involving grades still survive in some few examinations.

Ex. Express $46° \, 6' \, 9''$ in centesimal measure.

Rule.—Express in decimals of a right angle (see § 2, Ex. 2). Then divide off in twos from the decimal point.

Thus ·51225 right angle = 51 grades 22 primes 50 seconds (centesimal), a result written thus : $51^g \, 22' \, 50''$.

5. **A protractor** is an instrument for measuring angles. It consists usually of a thin, semicircular disc of card, metal, or other material graduated along its circumference from 0° to 180°. To measure any angle, it is placed with its centre **O** at the vertex of the angle, and its base **OA** along one of the lines bounding the angle. The number opposite any point **P** gives the angle **AOP** measured in degrees.

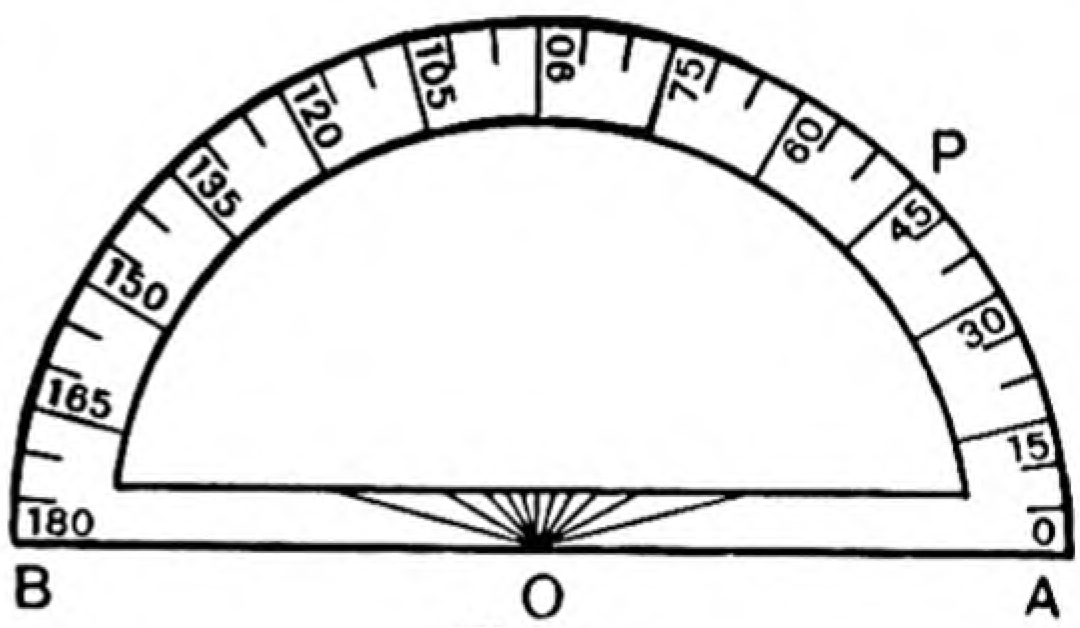


Fig. 1.

Since equal angles stand on equal arcs, the degree marks on the semicircle are equidistant.

It is thus easily inferred that the arc subtending, say, 10° is 10 times the arc which subtends 1°, and so on.

Hence *angles at the centre of a circle are proportional to the arcs which subtend them.*

[This important result is proved more fully in Euclid VI. 33.]

6. **Trigonometrical aspect of angles in general.**—Euclid's definition of an angle as " the inclination of two straight lines which meet " is hardly applicable unless the angle is less than two right angles.

In Trigonometry, an **angle** is therefore defined as *that which is described by a line which revolves about one of its extremities, in one plane, from one position to another.* When the straight line revolves about **O** from the position **OX** into the position **OP**, it is said to *describe the angle* **XOP**, and the revolving line is called the **radius vector** (Fig. 2).

The hands of a watch, or the spoke of a revolving wheel, exemplify this mode of describing angles. The minute hand of a watch describes a right angle or 90° in $\frac{1}{4}$ hour, 180° in $\frac{1}{2}$ hour, 360° in an hour. The hour hand describes 360° in 12 hours or 720° in a day, and so on; thus there is no limit to the magnitude of angle so described.

7. Let **O** be any fixed point, **OX** any fixed straight line through it. Draw the perpendicular line **OY**, and produce **XO**, **YO** backwards to **X′**, **Y′**. Imagine a straight line **OP** to start from the position **OX** and revolve about **O** in the direction of the arrow (Fig. 3), its extremity **P** describing a circle.

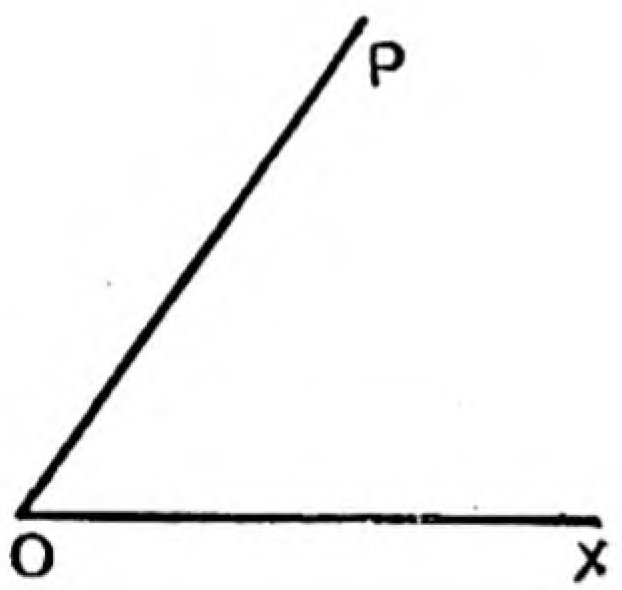


Fig. 2.

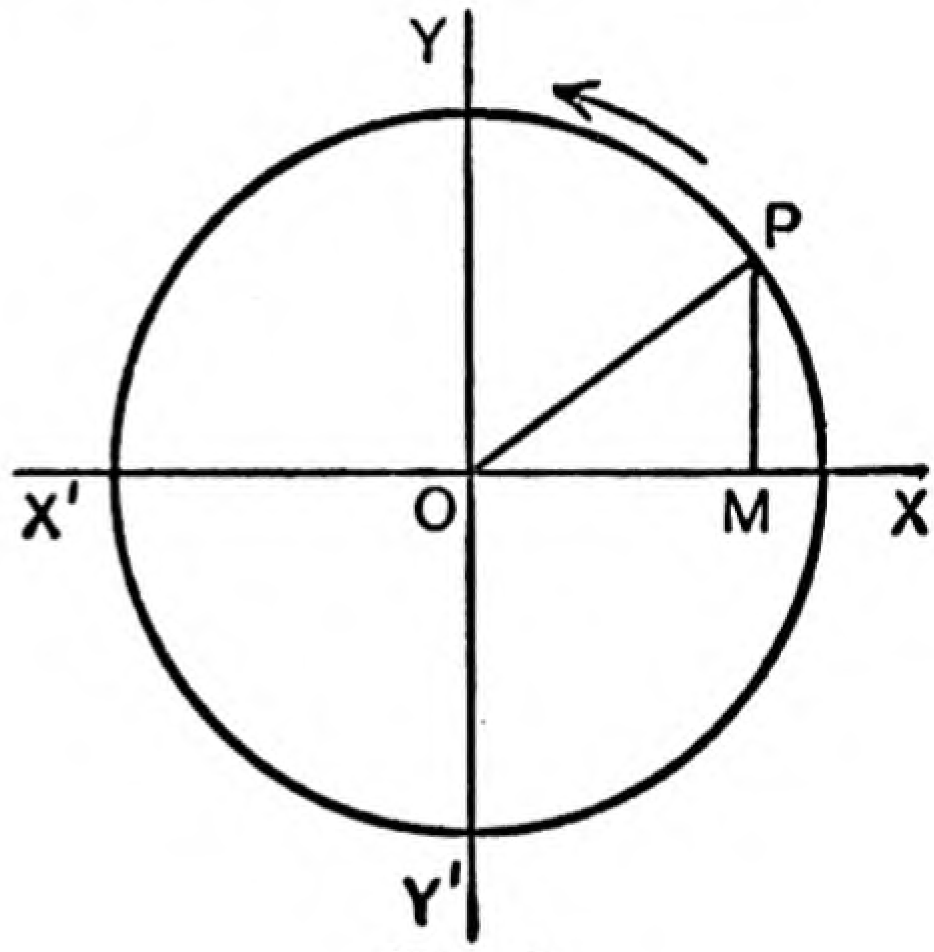


Fig. 3.

When it coincides with	**OY**,	it will have described	1 rt. angle or **90°**;
,, ,,	**OX′**,	,, ,,	2 rt. angles or **180°**;
,, ,,	**OY′**,	,, ,,	3 rt. angles or **270°**;
,, ,,	**OX** again,	,, ,,	4 rt. angles or **360°**.

The line will now be in the same position as at starting, although the angle described is 360°. If it continues to revolve, it will pass through the same positions as in the first revolution, and

when it again coincides with	**OY**	it will have described	360° + 90°, or **450°**;
,, ,,	**OX′**	,, ,,	360° + 180°, or **540°**;
,, ,,	**OY′**	,, ,,	360° + 270°, or **630°**;
,, ,,	**OX**	,, ,,	360° + 360°, or **720°**;

and so on, an additional 360° being described in each complete revolution.

If the revolving line **OP** stops between	**OX** and **OY**,	the angle **XOP** is said to lie in the	first quadrant;
,,	**OY** and **OX′**,	,,	second ,,
,,	**OX′** and **OY′**,	,,	third ,,
,,	**OY′** and **OX**,	,,	fourth ,,

Ex. What angles will the hour and minute hands of a clock describe between 12 o'clock and the instant when they are next together?

The hands are together at 12 o'clock; therefore they will be again together when the minute hand has described one revolution or 360° more than the hour hand.

Now in one hour the minute hand describes 360° and the hour hand 30°; hence the minute hand gains 330°;

∴ it will gain 360° in 1 hour × 360/330, that is, $1\frac{1}{11}$ hours.

Therefore the hands will be together at $1\frac{1}{11}$ hours past 12, that is, at 1 h. 5 m. 27 s. (to the nearest second).

In this time the hour hand will have described $30° \times 1\frac{1}{11}$, or 32° 43′ 38″, and the minute hand 360° more, or 392° 43′ 38″, fractions of 1″ being neglected.

EXAMPLES I.

1. How are angles measured? What fraction of a right angle is 5° 37′ 30″? What angle does the hour hand of a watch describe between 1 a.m. and 2.5 p.m.?

2. Define *degrees*, *minutes*, and *seconds*. Express $\frac{1}{32}$ of a right angle in degrees, minutes, and seconds. What angle does the hour hand of a watch describe between 2 a.m. and 3.25 p.m.?

3. At what time between 8 and 9 o'clock are the hands of a watch together?

4. Express 49·26° in degrees, minutes, and seconds.
   32·967° ,, ,, ,,
   85·6485° ,, ,, ,,

5. Reduce 42° 15′ 18″ to the decimal of a right angle.
   63° 19′ 17″ ,, ,, ,,
   4° 59′ 59″ ,, ,, ,,

6. The base of an isosceles triangle is 49° 9′ 9″: find the vertical angle.

7. The vertical angle of an isosceles triangle is as large again as either of the base angles: find the three angles.

8. The semi-sum of two angles of a triangle is 80°, and their semi-difference is 10°: find the three angles.

9. The angles of a right-angled triangle are in A.P.: find them.

10. If n be the number of sides of any rectilineal figure, the sum of its n angles is $(n - 2)\ 180°$.

11. Express in sexagesimal measure the angles of (i) a regular hexagon, (ii) a regular decagon, (iii) a regular quindecagon.

12. Show that the angles of a regular octagon and dodecagon are as $9 : 10$.

13. If an isosceles triangle be inscribed in a circle on the side of a regular inscribed heptagon as base, compare its vertical and base angles.

14. The angles of a pentagon are as the numbers 2, 3, 4, 5, 6 : find them.

15. Find the length of the arc which subtends an angle of 15° at the centre of a circle whose circumference is 18 ft.

16. Determine the difference in latitude of two places, one of which is a mile due N. of the other, if the circumference of the Earth be 25,000 miles.

17. The hour hand of a clock is 11° ahead of the minute hand between 3 and 4 p.m. : what is the exact time, and what will the time be when it is 22° behind ?

18. If the diameter of the Earth be 7,920 miles, and its circumference 25,000 miles, find the length of an arc on the sea which subtends an angle of 1′ at the centre of the Earth.

19. A bicycle wheel has 32 spokes : find the angle between each pair.

20. A carriage wheel has a circumference of 10 ft. : express, in degrees, the angle through which a spoke has turned while the wheel runs 7 yd.

21. A garden plot is laid out in the form of a regular decagon, and a man walks round the border of it starting from one corner : find the angle through which he must turn at every corner, and the whole angle through which he has turned when he comes back to his starting place.

22. The number of degrees in an angle is n times the number of minutes by which it is short of a right angle : find the angle in degrees.

23. If each angle of a regular polygon of $2n$ sides be to each angle of a regular polygon of n sides in the ratio $8 : 7$, find the angles of each polygon in degrees.

24. Find the number of sides in the regular polygon each of whose angles is 162°.

25. If the angle of a regular octagon were the unit of angular measurement, what would be the measure of an angle of 70° ?

26. There are two equilateral and equiangular polygons, one of which has twice as many sides as the other, and its angles half as large again. Find the number of degrees in the angle of each polygon.

*27. One circle rolls upon another of twice its radius : through what angle will it have turned round its own centre when it has gone twice round the other ?

28. Three cog-wheels work together ; the smallest has 24, the next 30, the third 60 cogs. Find through what angle the third will turn when the first has gone round once.

*29. Express in centesimal measure $49^\circ\ 43'\ 30'$, $23^\circ\ 12'\ 8''$, and $18^\circ\ 57'\ 3''$.

*30. Express in degrees, etc., the following angles : $31^g\ 51^`\ 10^{``}$, $8^g\ 32^`\ 11^{``}$, and $14^g\ 35^`\ 16^{``}$.

*31. Compare the angles $2^\circ\ 12'\ 18''$ and $2^g\ 45^`$, *i.e.* find the ratio of one to the other.

*32. Specify in which quadrants the following angles lie : 714°, 918°, 1821°, 2001^g.

*33. The number of degrees in an angle is less by 5 than the number of grades which it contains : find the angle in degrees.

CHAPTER II.

CIRCULAR MEASURE.

8. **The ratio of the circumference to the diameter of a circle.**—We are now about to introduce another mode of measuring angles. As this will involve considerations of the ratio of the circumference to the diameter of a circle, we shall first require to explain what is meant by this ratio, and how it is calculable.

If we were to take any circular object—*e.g.* a circular disc of cardboard, or the circular rim of a bowl—and if we measured its diameter across and also measured its circumference round with a tape, we should find that the circumference was a little over 3 times the diameter. If we were to measure off a length of 7 times the circumference, we should find this length to be almost exactly—but, if anything, just short of—22 times the diameter. Hence we conclude that the circumference of a circle is more than 3 times, and just short of $\frac{22}{7}$ times, the diameter. This we express by saying that

The **ratio** of the circumference of a circle to the diameter is greater than 3 : 1, and very nearly, but not quite, equal to 22 : 7.

In Trigonometry it is necessary to know the relation between the circumference of a circle and its diameter with greater accuracy than could be attained by actual measurement. The following article sketches out *one* way in which this ratio *could* be calculated from theoretical considerations to any degree of approximation. Better methods have been devised, but they belong to the higher applications of Trigonometry; it is sufficient here to prove that the calculation is possible.

9. The circumference of a circle can be calculated in terms of its diameter to any number of places of decimals.

Take any circle, and let its radius be r, and suppose it is required to find the circumference (*e.g.*) to 5 places of decimals.

Inscribe a regular hexagon in the circle. By Euclid IV. 15, each side of the hexagon is equal to the radius r. Hence the perimeter (*i.e.* the sum of the sides) of the hexagon is $6r$ or 3 times the diameter.

Bisect each of the arcs on which the sides of the hexagon stand, and join the points of bisection to the vertices. We thus obtain a 12-sided figure, and this approximates more nearly to the shape of the circle than the hexagon. By bisecting each of the arcs again, we obtain a new polygon with twice as many sides as the last, and by continually repeating the process, we obtain a series of polygons each of which approaches more closely to the circumference than the preceding one.

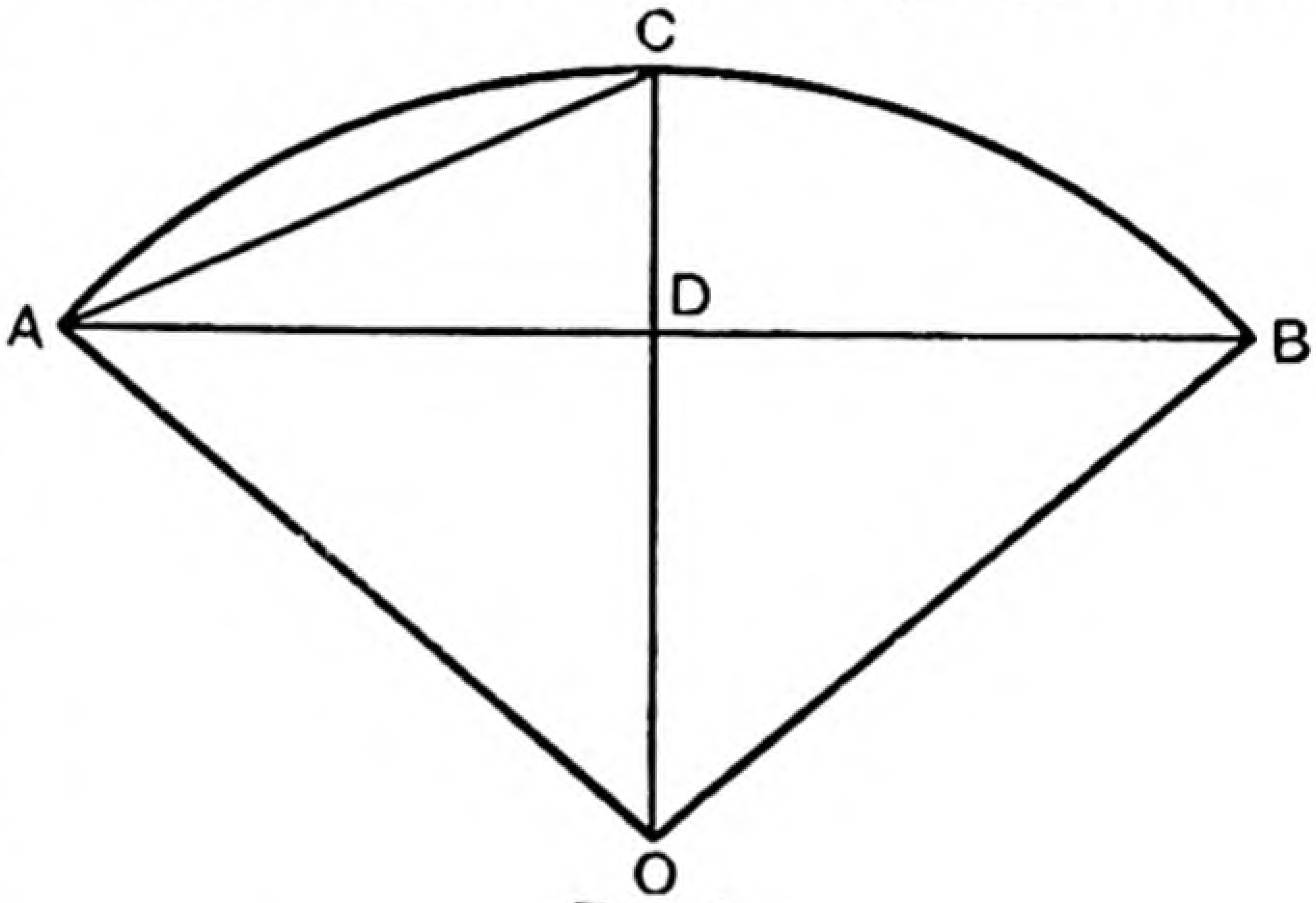


Fig. 4.

Knowing the perimeter of any one of these polygons, that of the next polygon in the series may be found.

For, if **AB** be the side of any inscribed polygon, **C** the middle point of the arc **AB**, then the side **AB** and the radius **OA** are known; hence the lengths **OD**, **DC**, and **AC** can be calculated by Euclid I. 47, and **AC**, the side of a polygon with double the number of sides of the original one, is thus known. Hence the perimeter of the latter polygon can be found. We may thus calculate in succession the perimeters of polygons of 12, 24, 48, 96, 192, . . . sides inscribed in the circle.

But when the process is carried far enough it is found that the perimeters of all succeeding polygons agree to the first 5 places of decimals, their lengths being 3·14159 . . . times the diameter. We conclude that even if the inscribed polygon had an infinite number of sides, its perimeter would still be 3·14159 times the diameter, to 5 places of decimals, and hence we infer that the circumference of the circle is 3·14159 . . . times the diameter.

10. The foregoing construction leads up to the following definition:—

DEF.—The **length of the circumference of a circle** is the limit to which the perimeter of an inscribed polygon tends

when the number of sides is made infinitely large, each side becoming infinitely small.

11. The circumferences of circles are proportional to their diameters.

If we take two circles and inscribe in them regular polygons with the *same* number of sides, the triangles (such as **OAB**, Fig. 5) which these sides subtend at the centres of the respective circles will be similar. Hence the sides of the polygons, and therefore their perimeters, are proportional to the radii, and therefore to the diameters, of the circles. By making the (equal) numbers of sides in the two polygons infinitely large, the same is seen to be true for the circumferences of the circles.

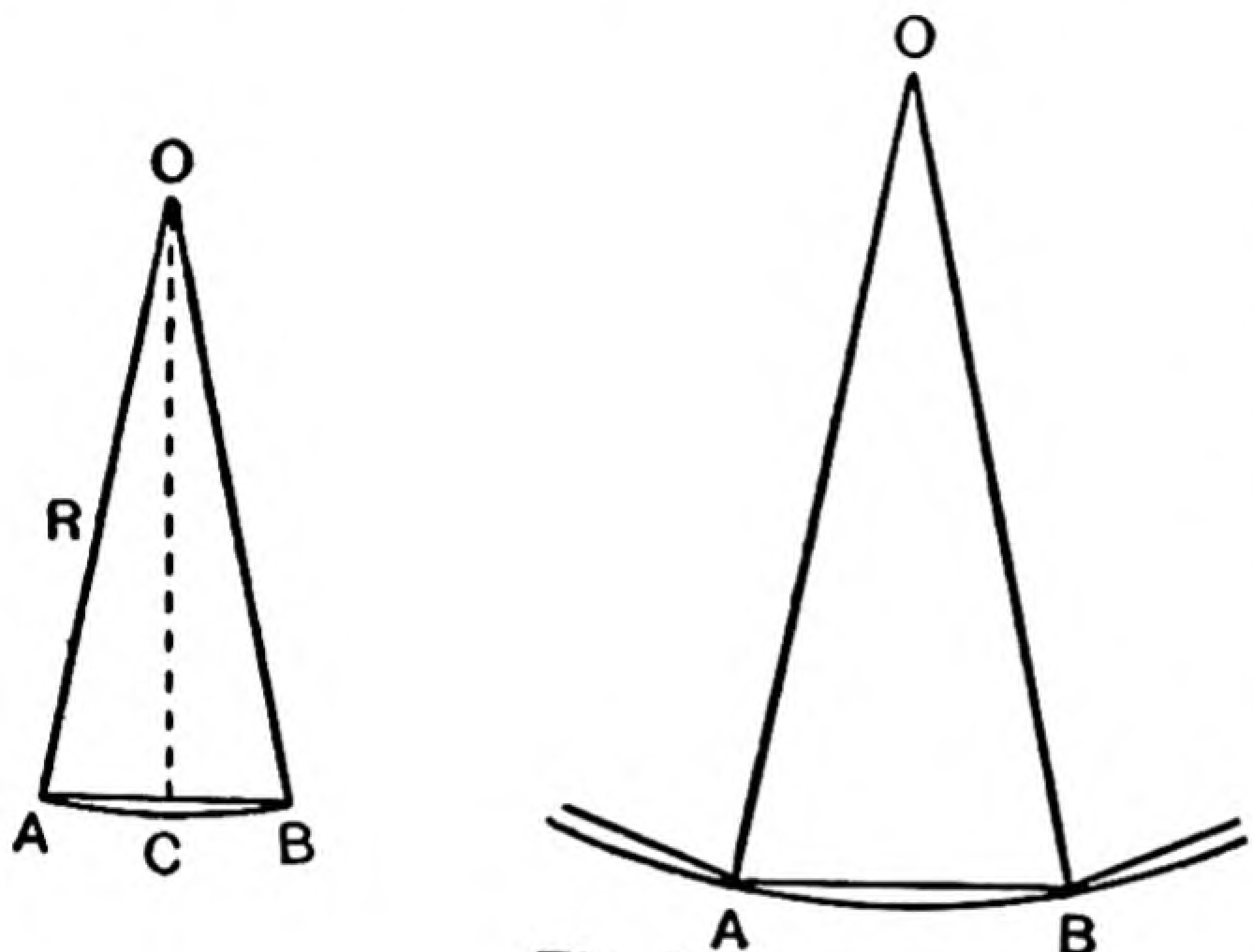


Fig. 5.

12. It thus follows that **the ratio**

$$\frac{\textbf{circumference}}{\textbf{diameter}}$$

is the same for all circles. As this ratio is constantly occurring in Trigonometry, it is convenient to represent it by a letter.

DEF.—*The Greek letter* π (Pi) *is always used to denote the ratio of the circumference of a circle to its diameter.*

Hence

$$\begin{aligned}\textbf{the circumference of a circle} &= \pi \textbf{ times its diameter}\\ &= 2\pi r, \text{ where } r \text{ is the radius} \qquad (2)\end{aligned}$$

Although the value of π can be calculated to any number of decimal places by various methods, such as that explained in the last article, the process never stops, and the figures never repeat themselves over

and over in the same order as they would do in an ordinary recurring decimal. Hence the value of π cannot be represented exactly by any arithmetical fraction, and this we express by saying that π is **incommensurable.**

Its value has, however, been calculated to 700 places of decimals. The result is of no interest except as a mere feat of arithmetical skill.

13. The value of π to 8 places is 3·14159265..., but for most purposes the common rough approximations to the value of π, viz.

$$\left.\begin{aligned}\pi &= \frac{22}{7} \text{ (= 3·1428..., correct to 2 decimal places only)}\\ \pi &= \mathbf{3{\cdot}1416} \text{ (correct to 4 places)}\end{aligned}\right\}\ldots(3)$$

are sufficient, and these alone need be remembered by the student.*

14. Circular Measure.

DEF. 1.—The angle subtended at the centre of a circle by an arc whose length is equal to the radius is called a **radian.**

DEF. 2.—The **circular measure** of an angle is the number of radians it contains.

Thus *the radian is the unit of circular measure,* so that the circular measure of the radian is 1.

An angle of one radian is sometimes denoted thus: 1^r or 1^c.

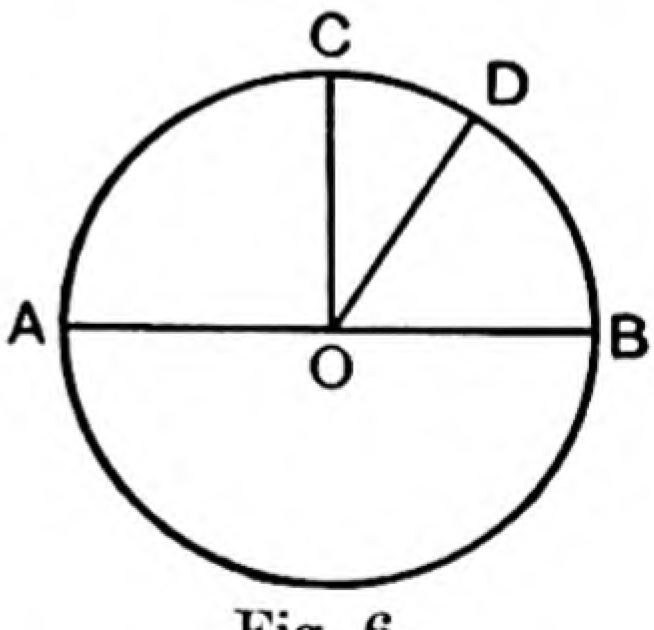


Fig. 6.

15. The circular measure of two right angles is π, **and hence the radian is an invariable angle.**

Let **ACB** be a semicircle, whose centre is **O** and whose radius is r. Along the circumference measure off the arc **BD** equal in length to the radius r. Then, by definition, **BOD,** the angle subtended by **BD,** is the radian; also the total angle subtended at **O** by the semicircular arc **BCA** is equal to 2 right angles.

* The value $355 \div 113 = 3{\cdot}141592|92 \ldots$ is correct to 6 places of decimals, and can easily be remembered by writing down the odd numbers 113355, and dividing the first three into the second three.

But angles at the centre of a circle are proportional to the the arcs on which they stand (§ 5), (Euc. VI. 33.)

$$\therefore \quad \frac{2 \text{ right angles}}{1 \text{ radian}} = \frac{\text{arc } \mathbf{BCA}}{\text{arc } \mathbf{BD}} = \frac{\frac{1}{2} \text{ circumference}}{\text{radius}}$$

$$= \frac{\frac{1}{2} . 2\pi r}{r} = \frac{\pi r}{r} = \pi; \qquad (\S\ 12)$$

$$\therefore \quad \textbf{2 right angles} = \boldsymbol{\pi} \textbf{ radians} \quad \ldots\ldots\ldots\ldots\ldots \quad \textbf{(4)}$$

or, the circular measure of 2 right angles $= \boldsymbol{\pi}$.

Again, $$\textbf{a radian} = \frac{\textbf{2 right angles}}{\pi} \ldots\ldots\ldots\ldots \quad \textbf{(4A)}$$

Hence the radian bears a constant ratio to a right angle which does not depend on the size of the circle used in the construction; in other words, *the radian is an invariable angle.*

16. The circular measure of any angle at the centre of a circle

$$= \frac{\textbf{arc subtending the angle}}{\textbf{radius of circle}}.$$

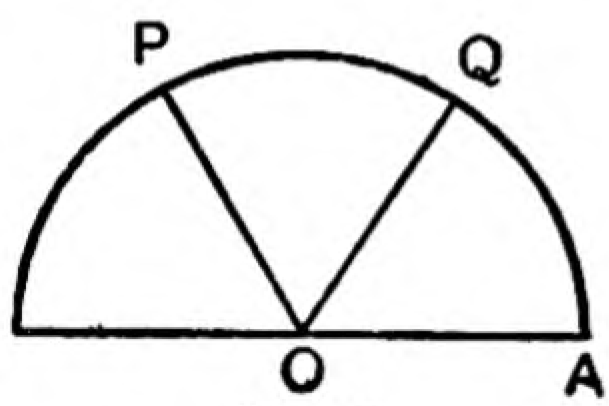


Fig. 7.

Let **AOP** be any angle subtended at the centre of a circle by the arc **AQP,** and let the arc **AQ** be measured off equal to the radius. Then, as in the last article, $\angle$**AOQ** is a radian. Hence

$$\frac{\angle \mathbf{AOP}}{\text{radian}} = \frac{\text{arc } \mathbf{AP}}{\text{arc } \mathbf{AQ}} = \frac{\text{arc } \mathbf{AP}}{\text{radius}};$$

$$\therefore \left.\begin{array}{l}\textbf{number of radians in } \angle \mathbf{AOP} \\ \textbf{or, circ. measure of } \angle \mathbf{AOP}\end{array}\right\} = \frac{\textbf{arc AP}}{\textbf{radius of circle}} \ldots \textbf{(5)}$$

17. To find the value of a radian in sexagesimal measure.

Since π radians $= 2$ rt. angles $= 180°$;

$$\therefore \quad 1 \text{ radian} = \frac{180°}{\pi} = \frac{180°}{3{\cdot}1416}$$

$$= 57{\cdot}296°$$

$$= \mathbf{57°\,17'\,45''}, \text{ or } \mathbf{206265''},$$

correct to the nearest second.

To find the circular measure of 1°.

$$180° = \pi \text{ radians};$$

$$\therefore \text{ circular measure of } 1° = \frac{\pi}{180} = \frac{3\cdot1416}{180} = 0\cdot01745 \ldots \text{ radians.}$$

18. **To transform from one system of angular measurement to the other,** we have the equation

$$\frac{D}{180} = \frac{C}{\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

where D, C are respectively equal to the numbers of degrees and radians in the angle.

Proof.—For each of these fractions

$$= \frac{\text{the angle}}{2 \text{ right angles}}.$$

It is necessary to reduce minutes and seconds to decimals of a degree before applying this rule.

Ex. To find the circular measure of 15°.

$$\frac{15}{180} = \frac{C}{\pi}; \quad \therefore \ C = \frac{\pi}{12};$$

$$\therefore \ 15° = \tfrac{1}{12}\pi \text{ radians, or, for brevity} = \tfrac{1}{12}\pi.$$

19. **Caution.**—Where the factor π occurs in the circular measure of an angle, as in the above example, it is usual to leave the result in this form, and not to evaluate it by substituting an approximate value for π. This is the case with most angles occurring in Trigonometry, and as the incommensurable number π is thus generally associated with the *circular measure* of an angle, it has become customary to omit specifying the unit of measurement of angles involving π. Thus π has come to have a double meaning; *when speaking of angles* it stands for two right angles; but in all other cases it represents the number 3·14159 . . ., never the number 180.

The reader should, at first at any rate, mentally supply the word "radians" when expressing an angle in terms of π, and should read the statement "$180° = \pi$" as "$180° = \pi$ radians."

Where the measures of angles are expressed by single algebraic letters, it is *usually* convenient to use letters of the Greek alphabet (α, β, . . . θ, etc.) to denote *circular measures*.

Illustrative Exercises.

1. Write down the following angles in degrees: $\frac{1}{6}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, π, $\frac{3}{2}\pi$, 2π.

2. Write down the circular measures of 30°, 60°, 120°, 180°, 360°, 45°, 90°, 135°, 270°.

3. Criticise the following reasoning : "$\pi = 180°$, therefore the circumference of a circle is 180 times the diameter."

4. Name the fourth proportional to the circumference of a circle, the diameter of the circle, and two right angles.

EXAMPLES II.

1. Assuming the numerical value of π to 5 decimal places, calculate to the nearest integer the number of minutes in the angle subtended at the centre of a circle by an arc of length equal to that of the radius of the circle.

2. Prove that, to turn circular measure into seconds, we must multiply by 206265.

3. Prove that, to turn seconds into circular measure, we must multiply by ·00000485.

4. If an angle be $\frac{2}{3}$ in circular measure, what is it in degrees, minutes, and seconds ?

5. What is the numerical value of a right angle in circular measure ?

6. Find the circular measure of an angle of 112° 43′.

7. Find the length of an arc which subtends an angle of 112° 43′ at the centre of a circle whose radius is 153 ft.

8. Find, in circular measure, the numerical value of an angle which in sexagesimal measure is 37° 15′.

9. The radius of a circle being 105 ft., find the length of the arc which subtends an angle of 37° 51′ at the centre.

10. What is the difference of latitude of two places, one of which is due N. of the other, at a distance of 30 miles from it, the radius of the Earth being taken as 4,000 miles ?

11. The driving-wheel of a railway engine is 7 ft. in diameter : how many revolutions will it make in a journey of 100 miles ?

12. If the radius of a circle be 25 ft., find the length of an arc which subtends 3″ at the centre.

13. If an observer cannot distinguish marks on a graduated circle closer together than $\frac{1}{20}$ in., what must be the least radius of the circle in order to measure angles of 1″ ?

14. What must the radius and circumference of a globe be in order that, when places are accurately mapped out on it, their distances may be on the scale of $\frac{1}{10}$ in. to the mile ? [Earth's radius = 3,960 miles.]

15. A piece of string is stretched on the above globe, of 5 ft. radius, at the Equator, between two places whose longitude differs by 10°.

The string measures 10·472 in. Calculate the ratio of the circumference to the diameter of a circle.

16. How many revolutions per minute are made by the wheel of a bicycle travelling 5 miles an hour, if its radius be 1 ft. 3 in. ?

17. A train is travelling at the rate of 40 miles an hour along a circular curve whose radius is 3 miles. Through what angle will it appear to have passed in 15 sec. to an observer stationed at the centre of the circle ?

18. The minute hand of a clock is 8 in. long, and the hour hand is only 6 in. Find through what space the point of each has moved in a quarter of an hour. How far apart will the points be at 12 o'clock, and when will the hands be next at right angles ?

19. Express in circular measure $6° 7' 8''$; and find to the nearest second the angle whose circular measure is ·7.

20. Express in circular measure, and also in degrees, the angle of a regular nonagon.

21. Of what angle is 1·5708 the circular measure ?

22. The angles of a triangle are to one another in the ratio 2 : 3 : 4 ; express them in circular measure and in degrees.

23. If 100 be the measure of a right angle, what would be the measure of an angle whose circular measure is $\frac{1}{3}$?

24. Express in circular measure the angle subtended at the centre of a circle, radius $3/\pi$, by an arc of length $\pi/3$. Express the same in degrees, etc.

25. Find the number of radians in the angles of a triangle which are in Arithmetical Progression and the greatest of which is 105°.

26. The perimeter of a certain sector of a circle is equal to half that of the circle of which it is a sector. Find the circular measure of the angle of the sector.

27. If it be found that the angle subtended at the centre of a circle by an arc equal to the radius is $57\frac{21}{71}$ of a degree, find the value of π.

28. What would be the measure of one radian if a right angle were taken as the unit of angular measurement ?

29. Two regular polygons of m and n sides have their angles in the ratio $n : 2m$. If m be $\frac{2}{3}n$, find the angles of each in circular measure.

30. If a_1, a_2, a_3 be the angles subtended by the arcs l_1, l_2, l_3 at the centre of the circles whose radii are r_1, r_2, r_3, show that the angle subtended by the arc $l_1 + l_2 + l_3$ at the centre of the circle whose radius is $r_1 + r_2 + r_3$ will be $\dfrac{a_1r_1 + a_2r_2 + a_3r_3}{r_1 + r_2 + r_3}$.

31. On the sexagesimal system of measurement, the measure of an angle in degrees exceeds its circular measure by unity. What is the magnitude of the angle expressed in these two ways of measurement ?

32. The angles of a triangle are such that the first is double the second, and the circular measure of the second is to the number of degrees in the third as π is to 270: find the number of degrees in each angle.

33. The angles of a triangle are in Arithmetical Progression, and the number of degrees in the least is to the circular measure of the greatest as 60 is to π: find the angles.

34. What is the numerical value of two right angles in circular measure?

35. If one of the acute angles of a right-angled triangle be 1·2 radians, what is the numerical value of the other acute angle (*a*) in circular measure, (*b*) in sexagesimal measure?

36. Find the length of an arc on the sea which subtends an angle of $2' \, 30''$ at the centre of the Earth, supposing the Earth to be a sphere of radius 4,000 miles. ($\pi = 3{\cdot}14159$.)

37. Express in degrees, grades, and circular measure the angle of a regular octagon.

38. The minute hand of a clock is 3 ft. long: how far will its extremity move in a quarter of an hour?

39. The diameter of the Sun is 883,220 miles: what is its circumference?

40. A mill sail, whose length is 20 ft., makes 10 revolutions per minute. Supposing its extremity goes half as fast as the wind, find the velocity of the wind in miles per hour.

41. One angle of a triangle is 2 radians, and another is $10°$: find the third to the nearest second.

42. If a third of a right angle were taken as unit angle, what would be the measure of $\pi/3$ radians?

43. Find the circular measure of the angle of a regular polygon of n sides.

44. Show that, if θ be the circular measure of any angle at the centre of a circle of radius r, the length s of the arc which it subtends is θr.

45. Express in circular measure the angle described by the hour hand of a clock in a minute and in a second of time.

46. Assuming that at a great distance a very small height may be considered as an arc of a circle whose centre is at the observer's eye, find the height of a column which at the distance of a mile subtends an angle of $1'$ at the eye.

47. What is the actual error in using the approximations for π given in § 13 in the case of a circle of 100 miles radius?

48. A train is running round a circular curve of 5 miles radius, and a man stationed at the centre of the curve finds that the minute hand of

his watch, once pointed to the train, exactly keeps pace with it: find the rate at which the train is going.

49. If the distance between the centres of the Earth and Moon is 60 times the Earth's radius, find the angle which the radius of the Earth subtends at the Moon's centre.

50. The apparent diameter of the Moon is 30′: find how far from the eye a coin of $\frac{1}{2}$-in. radius must be held so as just to hide the Moon's disc.

51. Calculate π to 2 decimal places, on the assumption that an angle of 200° at the centre of a circle subtends an arc of $3\frac{1}{2}$ radii.

52. If the radius of the Earth be 3,966 miles, find, to the nearest mile, the length of arc on its surface corresponding to each degree of latitude.

53. What must be the diameter of a bicycle wheel so that the number of revolutions in 10 sec. may indicate roughly the rate at which it is being ridden in miles per hour ?

54. If the Earth travels round the Sun in a circle whose radius is 95,000,000 miles, find the speed at which the Earth moves in miles per second, taking the year as 365 days.

CHAPTER III.

INTRODUCTION TO TRIGONOMETRIC FUNCTIONS.

20. In this chapter we shall introduce certain ratios connected with angles, and shall explain their meaning and use when the angles involved are *acute*. Owing to this limitation, the definitions given in the present chapter must not be regarded as final, but rather as leading up to the more general definitions of the next chapter.*

21. **Preliminary considerations.**—In § 1, we stated that *Trigonometry* means the measurement of triangles; now the simplest triangles to start with are right-angled triangles. Let **OPM, OQN** be two triangles, right-angled at **M, N,** and having their angles at **O** common; then, by Euclid I. 32, the remaining angles at **P, Q** are also equal; hence the triangles are equiangular to one another. Now it is proved in Euclid VI. 4 that such triangles have the sides of one proportional to the sides of the other, that is,

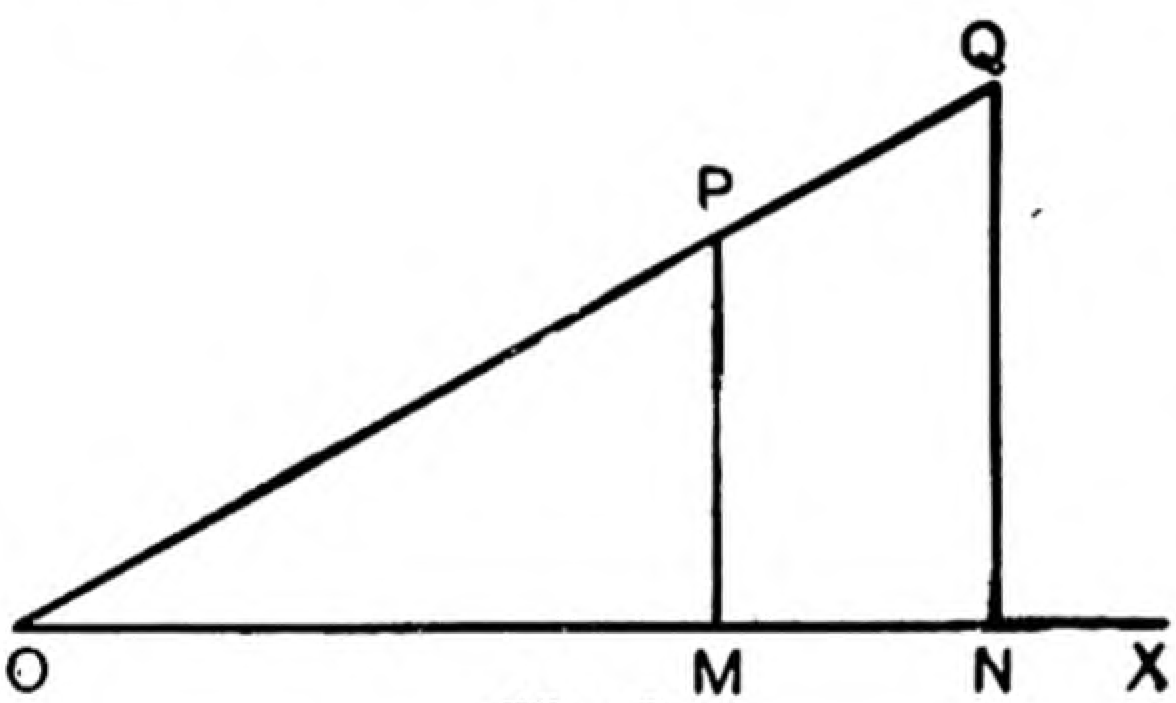


Fig. 8.

$$\frac{MP}{OP} = \frac{NQ}{OQ}, \qquad \frac{OM}{OP} = \frac{ON}{OQ}, \qquad \frac{MP}{OM} = \frac{NQ}{ON}.$$

Hence, if we know the ratios **MP/OP, OM/OP, MP/OM** for any right-angled triangle **OPM,** we know the ratios of the corresponding sides of any other right-angled triangle **OQN** which has its angle **NOQ** the same as **MOP.** *These ratios, therefore, depend only on the magnitude of the angle* **MOP,** and not on the size of the triangle. For this reason, tables can be

* For the same reason, we here consider only three out of the six trigonometric functions of an angle.

constructed giving the values of the ratios for different angles, and they have received names in accordance with the following definitions:—

22. The sine, cosine, and tangent of an angle.

DEF.—Let **XOQ** be any angle A. Take any point **P** on the line **OQ** bounding the angle, and complete the right-angled triangle **OPM** by drawing **PM** perpendicular on **OX**.

Then the ratio

$\frac{\mathbf{MP}}{\mathbf{OP}}$ is called the **sine** of the angle A and is written **sin** A,

$\frac{\mathbf{OM}}{\mathbf{OP}}$,, ,, **cosine** ,, ,, ,, **cos** A,

$\frac{\mathbf{MP}}{\mathbf{OM}}$,, ,, **tangent** ,, ,, ,, **tan** A,

all these ratios being called **trigonometric functions**, or **trigonometrical ratios, of the angle** A.

The triangle **OPM** is sometimes called a **triangle of reference**, or **fundamental triangle**, for the angle A. But the above ratios are considered as trigonometric functions of the *angle* A, not of the *triangle*.

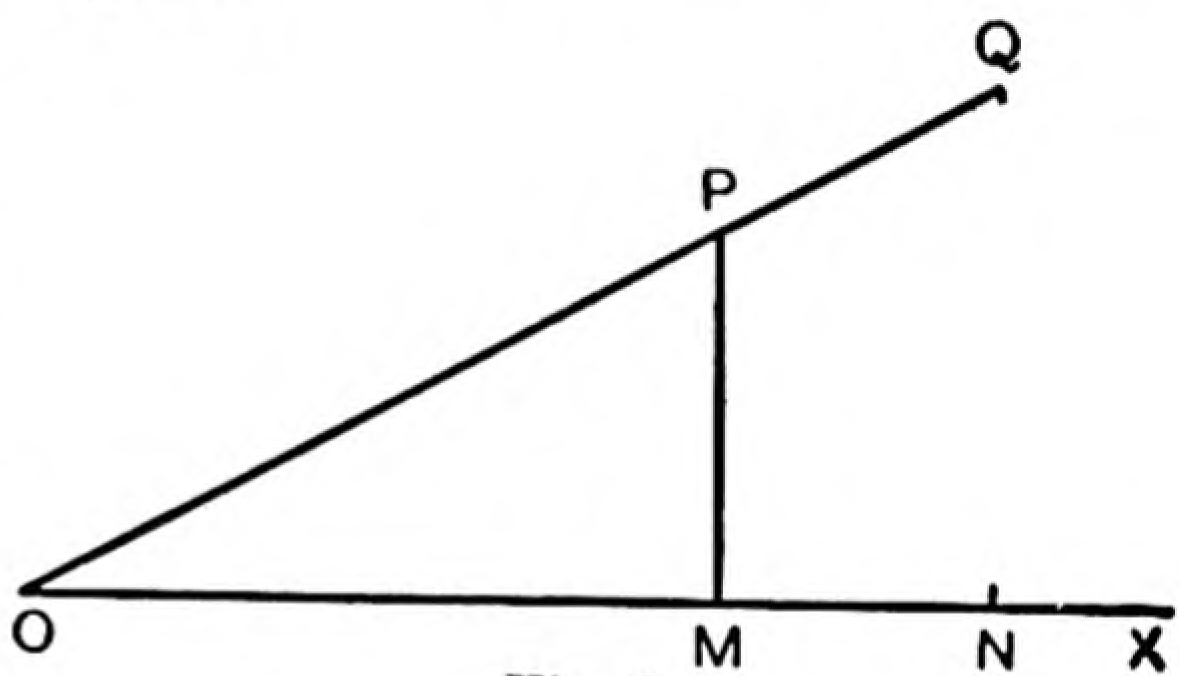


Fig. 9.

23. To remember these functions, the student may find it convenient to call **MP** (the side opposite $\angle A$) the *perpendicular*, **OM** (the side adjacent to $\angle A$) the *base*, **OP** the *hypotenuse* of the triangle, and then

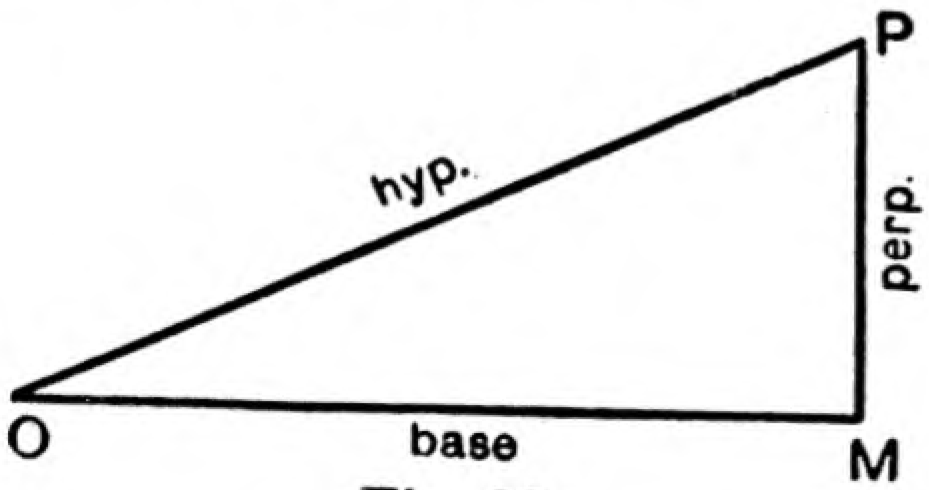


Fig. 10.

$$\sin = \frac{\text{perp.}}{\text{hyp.}},$$

$$\cos = \frac{\text{base}}{\text{hyp.}},$$

$$\tan = \frac{\text{perp.}}{\text{base}}.$$

Again, if **ABC** be the section of an inclined plane whose inclination to the horizon is A, then $\sin A =$ height/length, $\cos A =$ base/length, and $\tan A =$ height/base.

Ex. **ABC** is a triangle right-angled at **A**, and having **AB** = 5 in., **AC** = 12 in.

(i) To find the values of sin **ABC**, cos **ACB**, tan **ACB**.

(ii) To find the length of perpendicular from **A** on **BC**.

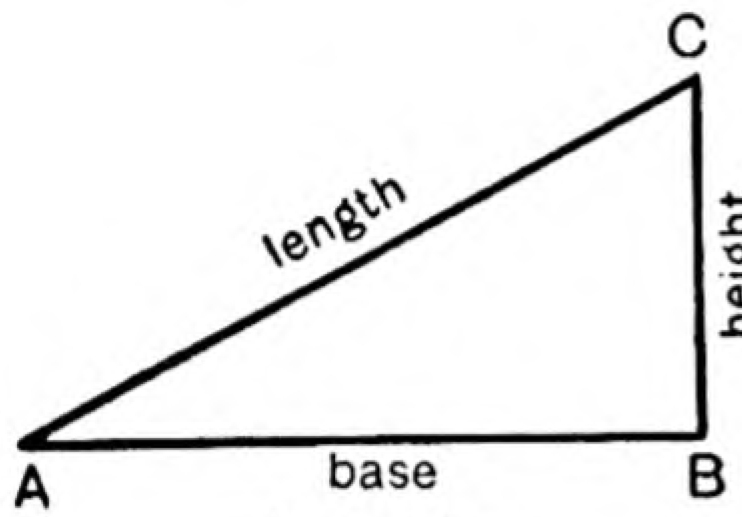


Fig. 10A.

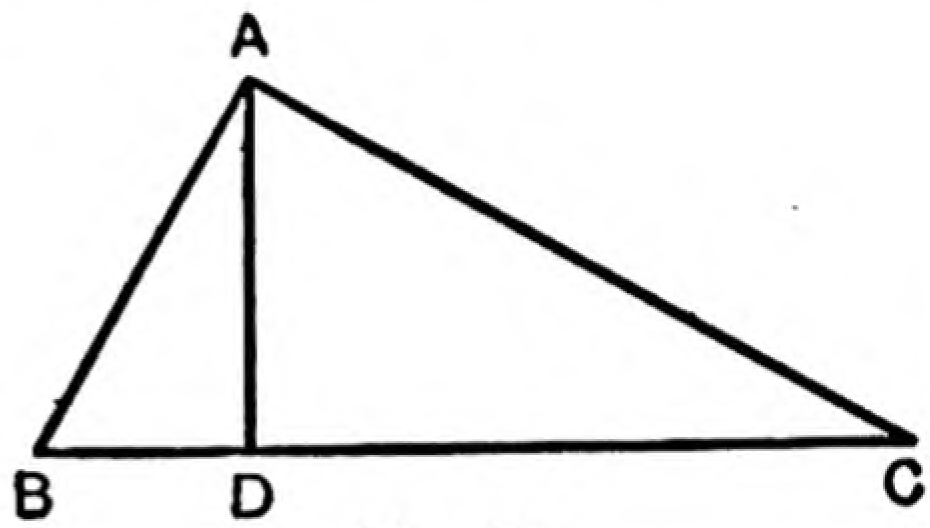


Fig. 11.

By Euclid I. 47,

$$\mathbf{BC}^2 = \mathbf{AB}^2 + \mathbf{AC}^2 = 5^2 + 12^2 = 25 + 144 = 169;$$

$$\therefore \quad \mathbf{BC} = 13 \text{ in.};$$

$$\therefore \quad \sin \mathbf{ABC} = \frac{\mathbf{AC}}{\mathbf{BC}} = \tfrac{12}{13}.$$

Again, since **CA** is the side adjacent to the angle **ACB**,

$$\therefore \quad \cos \mathbf{ACB} = \frac{\mathbf{CA}}{\mathbf{CB}} = \tfrac{12}{13},$$

and

$$\tan \mathbf{ACB} = \frac{\mathbf{AB}}{\mathbf{CA}} = \tfrac{5}{12}.$$

(ii) Let **AD** be drawn perpendicular on **BC**.

Then in the right-angled triangle **ADB** we know **AB** and want to find **DA**. But, by definition,

$$\frac{\mathbf{DA}}{\mathbf{BA}} = \sin \mathbf{ABD} = \tfrac{12}{13} \qquad \text{(from above);}$$

$$\therefore \quad \mathbf{DA} = \tfrac{12}{13}\,\mathbf{BA} = \tfrac{12}{13} \text{ of } 5 \text{ in.} = \tfrac{60}{13} \text{ in.} = 4\tfrac{8}{13} \text{ in.}$$

24. **Table of Trigonometric Functions.**—The numerical values of the sine, cosine, and tangent of multiples of 5° up to 90° are given, correct to four places of decimals, in the table below.

This table should be referred to whenever these values are required in the solution of simple problems.

To attempt to learn or remember any of the values would be *worse than useless.*

A rapid glance at the table may, however, possibly assist the reader in forming an idea of the nature of trigonometric functions, and it is advisable to consider carefully what the various entries mean. Thus, *e.g.* the statement that the sine of 20° is ·3420 implies that, if in *any* right-angled triangle one of the acute angles is 20°, the "perpendicular," or side opposite that angle, is approximately ·342 times the hypotenuse.

Angle	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°
Sine	0	·0872	·1736	·2588	·3420	·4226	·5000	·5736	·6428	·7071
Cosine	1	·9962	·9848	·9659	·9397	·9063	·8660	·8191	·7660	·7071
Tangent	0	·0875	·1763	·2679	·3640	·4663	·5773	·7002	·8390	1·0000

Angle	50°	55°	60°	65°	70°	75°	80°	85°	90°
Sine	·7660	·8191	·8660	·9063	·9397	·9659	·9848	·9962	1
Cosine	·6428	·5736	·5000	·4226	·3420	·2588	·1736	·0872	0
Tangent	1·1918	1·4281	1·7320	2·1445	2·7475	3·7320	5·6713	11·4300	∞

25. **The trigonometric functions obtained graphically.**— The numerical values of the sine, cosine, and tangent given in the above table are calculated by means which it is outside the scope of this book to indicate. It is, however, possible to verify some of the results by accurate drawing.

Ex. Find graphically the values of the sine, cosine, and tangent of an angle of 40°.

With the aid of a protractor, construct an angle **AOB** of 40°, and mark a point **P** on **OA** so that **OP** = 10 cm. From **P** draw **PM** perpendicular to **OB**, meeting it in **M**.

Now measure the lengths **OM**, **MP** as accurately as possible. Thus

$$\mathbf{OM} = 77 \text{ mm.} = 7{\cdot}7 \text{ cm.}$$

$$\mathbf{MP} = 64 \text{ mm.} = 6{\cdot}4 \text{ cm.}$$

$$\therefore \quad \sin 40^\circ = \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{6{\cdot}4}{10} = {\cdot}64.$$

$$\cos 40^\circ = \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{7{\cdot}7}{10} = {\cdot}77.$$

$$\tan 40^\circ = \frac{\mathbf{MP}}{\mathbf{OM}} = \frac{6{\cdot}4}{7{\cdot}7} = {\cdot}83.$$

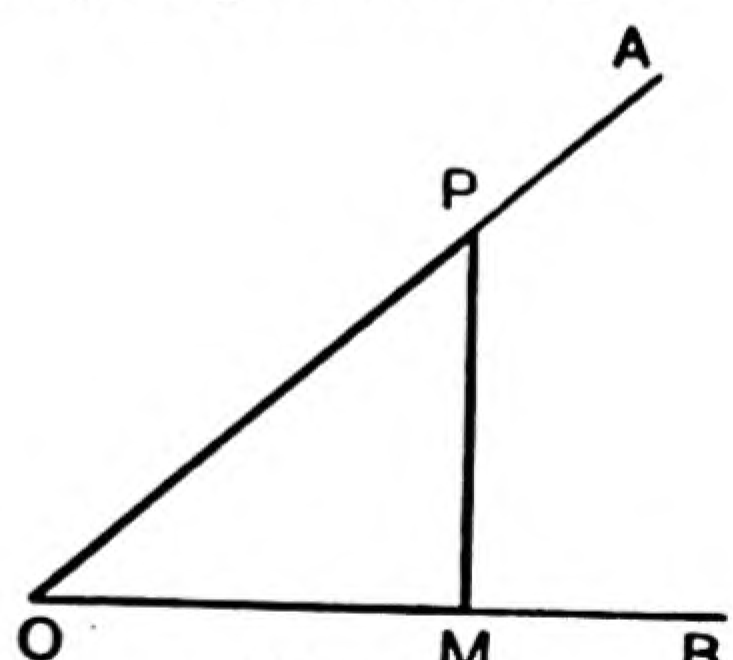


Fig. 12.

The student should verify for himself some of the other values in the above table.

26. **Solution of right-angled triangles.**—We may illustrate the trigonometrical notation by applying it to the solution of right-angled triangles of which some of the parts are given.

By **the six parts of a triangle** are meant its three sides and its three angles, or, more accurately, the *lengths* of its three sides and the *magnitudes* of its three angles. The letters **A, B, C** usually denote the angles, and a, b, c the sides *opposite* them: in this book A, B, C usually denote the *magnitudes* of the angles.

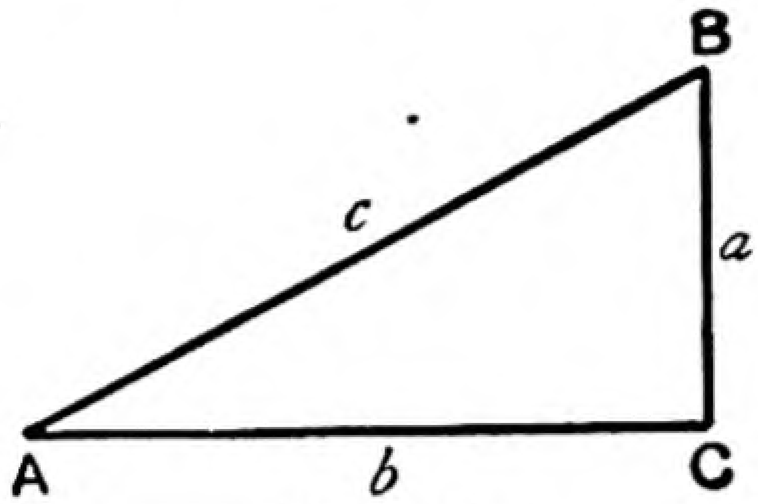


Fig. 13.

In right-angled triangles, it is usual to take **C** as the right angle; c will then denote the hypotenuse.*

If *three* of the parts of a triangle are given, of which *one at least is a side*, we may here state (without proof) that it is generally possible to determine the other three parts. This process is called **solving the triangle.** If the triangle is right-angled, the right angle is one of the given parts, and so we only require two other parts to be given—these may be either one side and one of the acute angles, or two of the three sides.

When one of the acute angles A, B is known, the other acute angle can be found from Euclid I. 32, which gives

$$A+B+C=180°;$$

whence, since $$C=90°,$$

$$A+B=90°; \quad \therefore \quad B=90°-A, \quad A=90°-B \quad \text{.......(7)}$$

When two of the three sides a, b, c are known, the third can be found from Euclid I. 47, which gives

$$c^2=a^2+b^2; \quad \therefore \quad b^2=c^2-a^2, \quad a^2=c^2-b^2 \quad \text{........(8)}$$

27. **The relations between the sides and angles** may be

* This notation is not the same as was used in § 22. It is important that the student should learn to write down the trigonometric functions of angles when the letters used in naming the triangle of reference are varied in every possible way.

found by writing down the definitions of the trigonometric functions of A, B in terms of the sides.* We thus obtain

$$\sin A = \frac{a}{c} \;\ldots\ldots\; (9\text{A}) \qquad \sin B = \frac{b}{c} \;\ldots\ldots\; (9\text{B})$$

$$\cos A = \frac{b}{c} \;\ldots\ldots\; (10\text{A}) \qquad \cos B = \frac{a}{c} \;\ldots\ldots\; (10\text{B})$$

$$\tan A = \frac{a}{b} \;\ldots\ldots\; (11\text{A}) \qquad \tan B = \frac{b}{a} \;\ldots\ldots\; (11\text{B}).$$

By clearing of fractions, these relations may also be written in the form—

$$a = c \sin A \;\ldots\ldots\ldots\; (9\text{A}) \qquad b = c \sin B \;\ldots\ldots\ldots\; (9\text{B})$$
$$b = c \cos A \;\ldots\ldots\ldots\; (10\text{A}) \qquad a = c \cos B \;\ldots\ldots\ldots\; (10\text{B})$$
$$a = b \tan A \;\ldots\ldots\ldots\; (11\text{A}) \qquad b = a \tan B \;\ldots\ldots\ldots\; (11\text{B}).$$

The relations (7) to (11B) are more than sufficient to solve the triangle when two parts are given besides the right angle, and we may select those formulae which are most convenient.

The following examples are principally intended to familiarise the reader with the use of trigonometric functions. A right-angled triangle can always be solved in a variety of ways, according to which part is determined first, and so on; and it will be found an instructive exercise to select different formulae (from 7–11B) for solving these examples, and to verify that the results are in every case the same.

Case I.—**Given one angle and a side.**

Ex. 1. Given $A = 70°$, $c = 250$ metres, find a, b.

Here, with reference to the angle A, a is the "perpendicular" and b the "base"; hence the proper relations to write down are

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c};$$

$$\therefore \quad a = c \sin A = 250 \sin 70°,$$
$$b = c \cos A = 250 \cos 70°.$$

From the table on p. 21,

$$a = 250 \times \cdot 9397 = 234 \cdot 925 \text{ metres},$$
$$b = 250 \times \cdot 3420 = 85 \cdot 5 \text{ metres}.$$

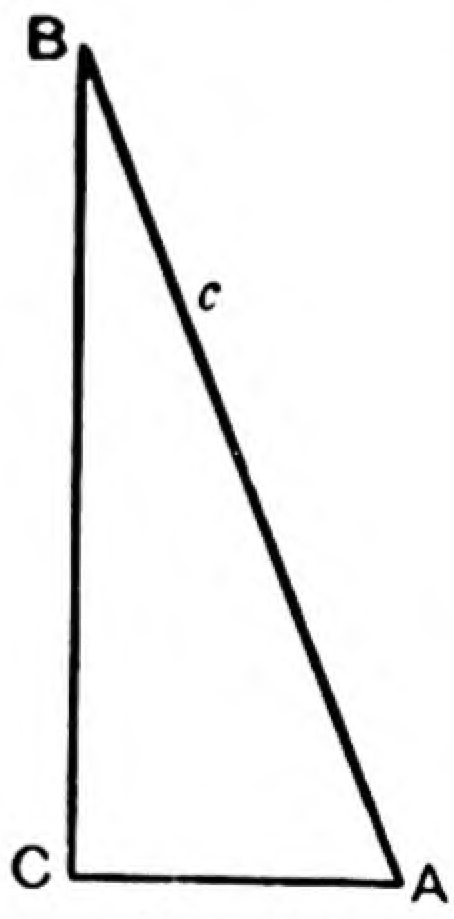


Fig. 14.

* The results should not be committed to memory; but, instead of doing so the student should practise writing them down from a figure.

Ex. 2. Given $A = 25°$, $b = 15$ ft., find a and c.

Here $$B = 90° - A = 65°.$$

Drawing a figure, we see that, with reference to $\angle A$, b is the "base" and a the "perpendicular," and the trigonometric function involving these two is the *tangent*. We therefore write down, from the definition,

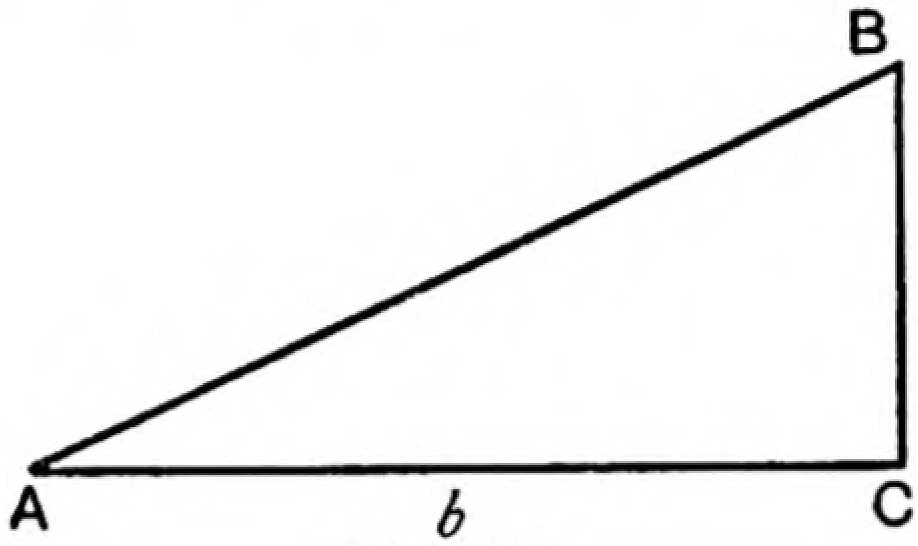


Fig. 15.

$$\tan A = \frac{a}{b}; \text{ whence } a = b \tan A;$$

$\therefore \quad a = 15 \tan 25° = 15 \times \cdot 4663$ (from table on p. 21)

$= 6\cdot 994$ ft.

We might now find c from*

$$c^2 = a^2 + b^2 = 15^2 + (6\cdot 994)^2.$$

The arithmetic is shorter if we work with the relation connecting the hypotenuse c with the base b. This relation is

$$\cos A = \frac{b}{c}, \quad \therefore \quad c \cos A = b, \quad \text{or} \quad c \cos 25° = 15.$$

From the table, we obtain $c \times \cdot 9063 = 15$, whence, by division, $c = 16\cdot 55$ ft.

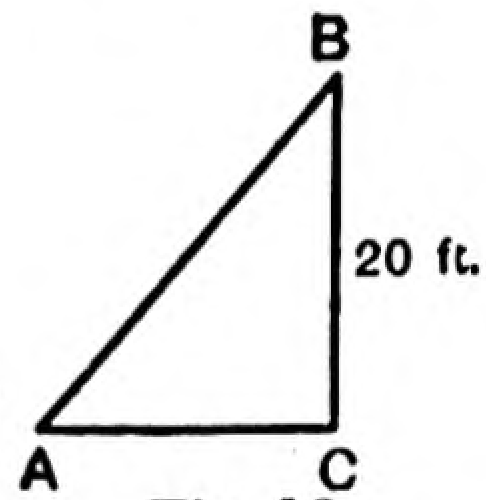


Fig. 16.

Ex. 3. Given $A = 50°$, $a = 20$ ft., find B and b.

Here $$B = 90° - 50° = 40°.$$

The trigonometrical ratios involving a and b are the tangents of A and B, and from the figure we write

$$\tan A = \frac{a}{b}, \quad \tan B = \frac{b}{a};$$

whence $$b = \frac{a}{\tan A} = \frac{20}{\tan 50°}, \quad \text{or} \quad b = a \tan B = 20 \tan 40°.$$

The second is the more convenient form to use, and with the table on p. 21 leads to

$$b = 20 \times \cdot 8390 = 16\cdot 78 \text{ ft.}$$

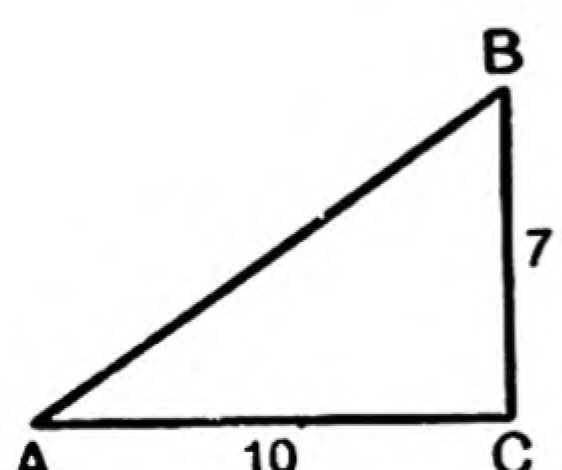


Fig. 17.

CASE II.—**When two side are given.**

Ex. 1. Given $a = 7$ ft., $b = 10$ ft., to find c, A, B.

Here $$c^2 = a^2 + b^2 = 149,$$
whence $$c = 12\cdot 206 \ldots \text{ ft.}$$

---

* The student should verify that this method leads to the same result as the following: viz. $c = 16\cdot 55$.

The trigonometric functions involving a, b are the *tangents* of A, B, and, from the figure,

$$\tan A = \frac{a}{b} = \tfrac{7}{10} = \cdot 7, \ \tan B = \tfrac{10}{7}.$$

Since $\tan 35° = \cdot 7002 = \cdot 7$, very nearly (from table)

$$A = 35°, \text{ approximately, and } B = 90° - 35° = 55°.$$

Ex. 2. Given $c = 50$ ft., $a = 17{\cdot}1$ ft., to find b, A, B.

Here $$c^2 = a^2 + b^2;$$

$$\therefore \quad b^2 = c^2 - a^2 = (c - a)(c + a) = (50 - 17{\cdot}1)(50 + 17{\cdot}1)$$
$$= 32{\cdot}9 \times 67{\cdot}1 = 2207{\cdot}59;$$
$$\therefore \quad b = 46{\cdot}98 \ldots, \text{ or very nearly 47 ft.}$$

Again, A may be found from the relation

$$\sin A = \frac{a}{c} = \frac{17{\cdot}1}{50} = \cdot 342 = \sin 20°; \qquad \text{(from table)}$$

$$\therefore \quad A = 20° \text{ and } B = 90° - 20° = 70°.$$

[Or we might have found B directly from the relation $\cos B = a/c = \cdot 342$; whence the table gives $B = 70°$.]

28. Practical Applications of Trigonometry.—The use of trigonometrical functions also enables us to solve many simple and useful problems in the measurement of heights and distances.

The following definitions will be required:—

Def.—When an object is observed from below, the angle which the line joining it to the observer makes with the horizon is called the **altitude,** or **angle of elevation,** of the object.

When the observer is higher than the object, the corresponding angle is called the **angle of depression.**

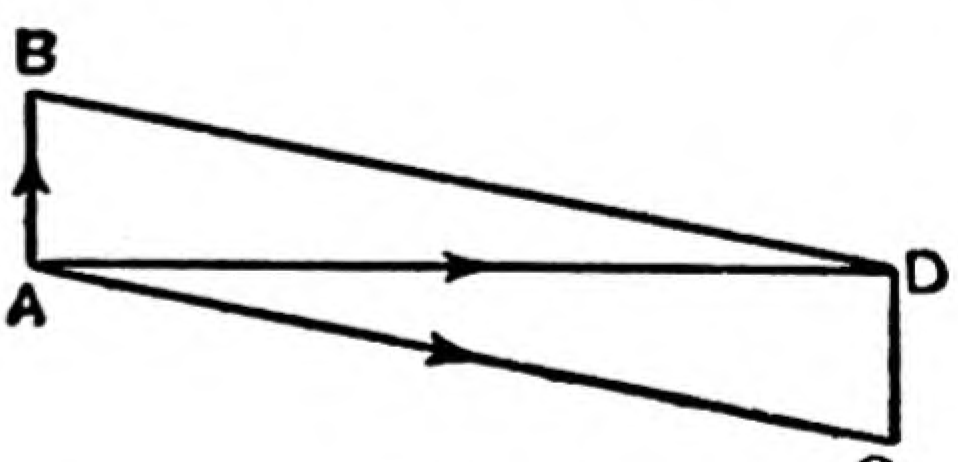


Fig. 18.

Thus, in Fig. 18, if **AD** be horizontal, $\angle$ **ADB** is the altitude or angle of elevation of **B** as seen from **D**, and $\angle$ **DAC** is the angle of depression of **C** as seen from **A**.

In actual observations, angles of elevation and depression can be determined by the use of an instrument called a *theodolite.*

In working problems on heights and distances, the figure may often be divided into right-angled triangles. This comes natural, since the *heights* are usually perpendicular to the *distances*.

Ex. 1. A hill rises at an inclination of 10° to the horizon. To find, in feet, the height risen in walking a mile up hill.

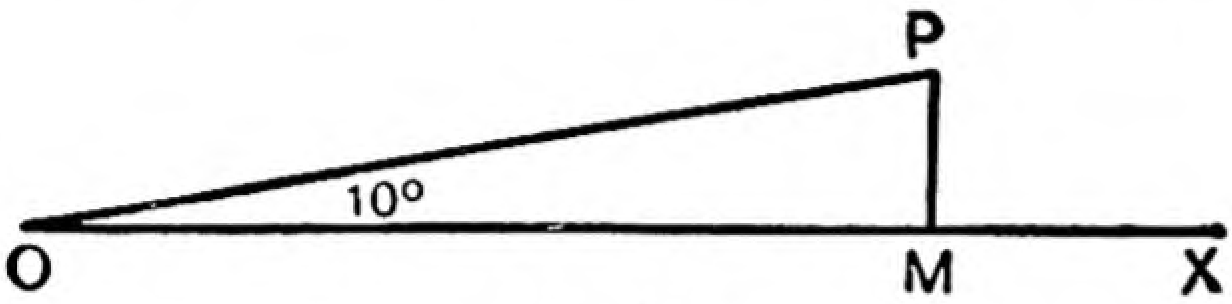


Fig. 19.

Let **OP** represent a mile of the hillside, and drop **PM** perpendicular on the horizontal line through **O**. We know **OP** and we want to find **MP**, having given $\angle$**MOP** $= 10°$. Since **MP**, **OP** are the "perpendicular" and "hypotenuse," the proper relation to take is

$$\frac{\mathbf{MP}}{\mathbf{OP}} = \sin \mathbf{MOP} = \sin 10° = \cdot 1736;$$

$\therefore$ **MP**, the vertical height risen $= \mathbf{OP} \times \cdot 1736 = 5280 \text{ ft.} \times \cdot 1736$
$= 917$ ft.,

correct to the nearest foot.

Ex. 2. From a point 430 ft. distant from the base of a tower, in a horizontal direction, the top is seen in a direction making an angle 13° with the horizon. To find the height of the tower, given

$$\tan 13° = \cdot 23.$$

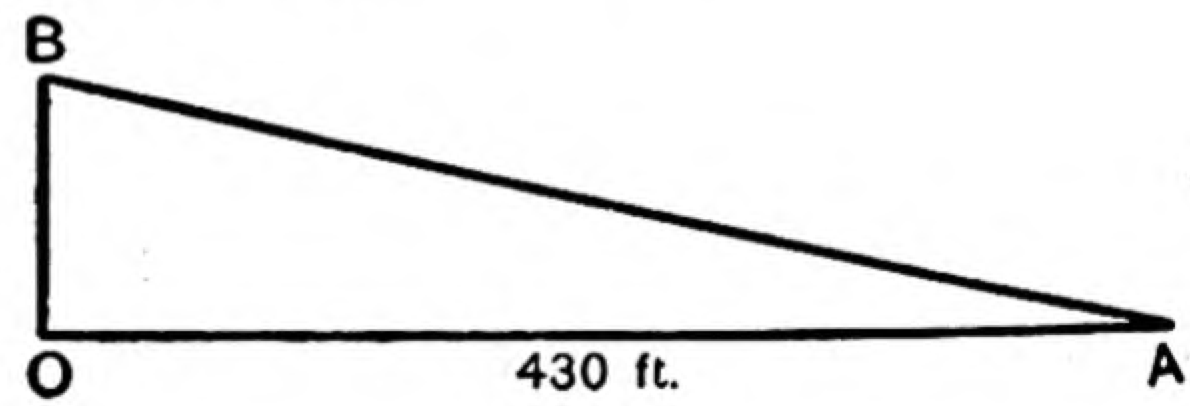


Fig. 20.

Let **OB** be the tower, **A** the position of the observer. Then we know

$$\mathbf{AO} = 430 \text{ ft.},$$

$$\angle \mathbf{OAB} = 13°,$$

and we want to find **OB**. Since **AO**, **OB** are the "base" and "perpendicular," considered with reference to the angle 13°, the proper ratio to take is

$$\frac{\mathbf{OB}}{\mathbf{AO}} = \tan 13° = \cdot 23;$$

$\therefore$ **OB**, the required height of the tower $= 430 \text{ ft.} \times \cdot 23$
$= 98{\cdot}9 \text{ ft.} = 99$ ft., nearly.

Ex. 3. To find (roughly) the altitude of the sun, having given that a stick 5 ft. high casts a shadow 6 ft. long on a horizontal plane.

Let **MP** represent the stick, **MO** the shadow. Then the line **OP** joining the extremities of the shadow and stick, when produced, passes

through the sun, and the $\angle$ **MOP** is called the *altitude* of the sun. Since **OM**, the "base," and **MP**, the "perpendicular," are known, the proper relation to take is

$$\tan \mathsf{MOP} = \frac{\mathsf{MP}}{\mathsf{OM}} = \tfrac{5}{6} = \cdot 833 \ldots;$$

$\therefore$ $\angle$ **MOP** the required altitude $= \angle 40°$, roughly (by table).

The student, in working such problems, will often find it useful to check his calculations by drawing a figure accurately to scale and obtaining the result by actual measurement.

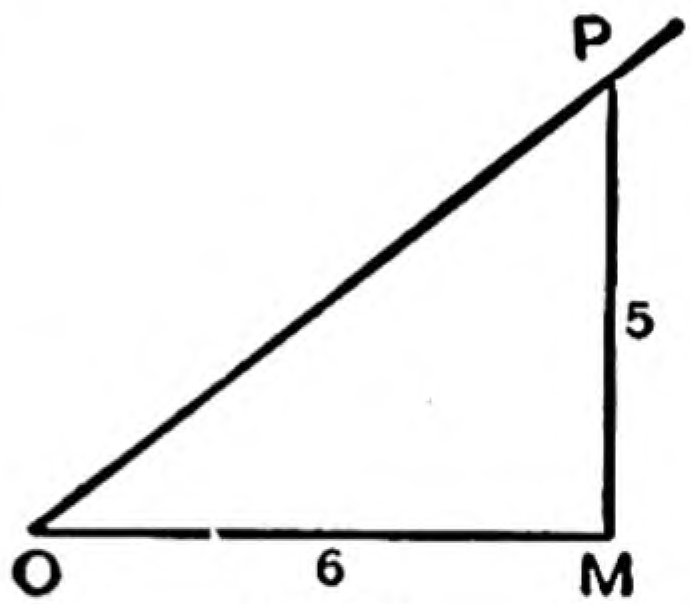


Fig. 21.

Thus, in Ex. 2, taking an inch to represent 50 ft., we draw **OA** of length 8·6 in. With the aid of a protractor, construct at **A** the angle **OAB** equal to 13°. At **O** draw **OB** perpendicular to **OA**, meeting **AB** at **B**. Measure **OB**.

The length of **OB** should be roughly 2 in. Since 2 in. represent 2×50 ft., this graphical method gives the height of the tower as 100 ft. (roughly).

Similarly, to check the result of Ex. 3, Fig. 21 should be drawn accurately to scale taking, say, 1 in. to represent 1 ft., and the angle **POM** measured with a protractor.

EXAMPLES III.

1. Find the tangents of the base angles of a triangle in which the lengths of the sides are 5 in. and 3·25 in., and the perpendicular from the vertex on the base is 3 in.

Solve the following triangles (2–22), using the table in § 24:—

2. $A = 20°$, $B = 90°$, $c = 80$.
3. $A = 5°$, $B = 85°$, $a = 100$.
4. $C = 90°$, $a = 10$, $c = 20$.
5. $A = 10°$, $C = 80°$, $b = 6$.
6. $A = 90°$, $C = 65°$, $c = 40$.
7. $B = 45°$, $a = 30$, $b = 30$.
8. $A = 35°$, $B = 55°$, $a = 1000$.
9. $A = 15°$, $B = 90°$, $c = 70$.
10. $B = 30°$, $C = 90°$, $b = 22$.
11. $C = 90°$, $a = 866$, $b = 500$.
12. $A = 90°$, $b = 1736$, $c = 9848$.
13. $B = 40°$, $C = 50°$, $b = 839$.
14. $A = 25°$, $B = 90°$, $a = 96$.
15. $B = 35°$, $C = 90°$, $b = 7$.
16. $A = 90°$, $C = 45°$, $a = 11$.
17. $A = 50°$, $B = 90°$, $c = 18$.
18. $A = 5°$, $B = 85°$, $c = 25$.
19. $A = 90°$, $a = 1250$, $c = 109$.
20. $C = 90°$, $b = 171$, $c = 500$.
21. $A = 90°$, $a = 2500$, $b = 855$.
22. Find B, a, and b in a triangle; given $c = 20$, $A = 30°$, $C = 90°$.

23. If from the top of a post a string twice its length be stretched tight to a point on the ground, what angle will the string make with the post?

24. From the table in § 24 (or otherwise) calculate to 4 places of decimals the values of the following trigonometrical expressions:—

(i) $\sin 50° - \cos 50°$, (ii) $(\sin 50°)^2 + (\cos 50°)^2$,

(iii) $\sin 55° \times \cos 25° - \cos 55° \times \sin 25°$; (iv) $\dfrac{\tan 75° - \tan 30°}{1 + \tan 75° \tan 30°}$;

and verify the statement that $\sin 65° = \tan 65° \times \cos 65°$.

25. The length of the string attached to a kite is 300 ft., and the kite's elevation is found to be 20°. Find the height of the kite from the ground. Check your result by an accurate drawing.

26. A ladder 13 ft. long stands against a wall, and makes with the ground an angle whose tangent is 2·1445. Find how far it reaches up the wall.

27. A rope 20 ft. long is fastened at one end to a point 20 ft. from the top of a flagstaff, and the other end is attached to a peg in the ground. It is found that the rope makes an angle of 60° with the ground. Find the height of the flagstaff.

28. A vein of coal is known to dip downwards in a straight line inclined at an angle of 20° to the horizon. How deep will a shaft have to be sunk in order to reach the vein from a spot on the surface situated a mile away from the place where the coal crops up to the surface?

29. A tower on the bank of a river is 120 ft. high, and the angle of elevation of its top from the opposite bank is 20°. Find the breadth of the river. Solve this problem also by drawing the figure.

30. What is the length of the shadow cast by a column 80 ft. high, when the sun's altitude is 70°?

31. Standing square in front of one corner of a house, I observe that its height subtends an angle whose tangent is 2, while its length subtends an angle whose tangent is 3. I then measure the length, and find it to be 30 ft. What is the height?

32. A pole being broken by the wind, its top struck the ground at an angle of 40°, and at a distance of 21 ft. from the foot of the pole. Find the whole height. Check your result by drawing a figure to scale.

33. The shadow of a church tower extends 56 yd. from its base. Find its height, it being observed that a 2-ft. rule at the same time casts a shadow 5 ft. long.

34. Sailing in company with another ship, I am ordered to keep at a distance of 1,000 ft. from her. Knowing that her mast reaches 87 ft. 6 in. above the level of my eye, what angle of elevation must the mast have?

35. The shadow of a tower 200 ft. high, standing 50 ft. from the bank of a river, falls straight across it and just reaches the opposite bank when the sun's altitude is 55°. Find the breadth of the river.

36. The shadow of a spire, standing 30 ft. from the bank of a river, falls straight across it and just reaches the opposite bank when the sun's altitude is 50°. The breadth of the river is 200 ft. Find the height of the spire.

37. A man 6 ft. high sees the top of a post at an elevation of 10° when standing at a distance of 100 yd. How far off must he go to see it at an elevation of 5°?

38. Construct an angle of $22\frac{1}{2}°$ either with a protractor or by geometrical methods, and, by measuring the sides of its fundamental triangle, calculate the values of its sine, cosine, and tangent to 2 decimal places.

39. Construct, either with a protractor or geometrically, an angle of 60° and, by measuring the sides of its fundamental triangle, calculate the values of its sine, cosine, and tangent to 2 decimal places.

40. Construct an angle of $37\frac{1}{2}°$ and, by measuring the sides of its fundamental triangle, calculate the values of its sine, cosine, and tangent to 2 decimal places.

CHAPTER IV.

GENERAL DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS.

29. **Positive and negative lines.**—Before we can define the trigonometric functions of angles greater than a right angle, it will be necessary to explain how lines measured in opposite directions can be distinguished by prefixing the algebraic signs $+$ and $-$ to the numerical measures of their lengths. We commence by giving the following example:—

Ex. A man starts from a place **B**, 9 miles east of a given town **A**, and walks due westwards at the rate of 3 miles an hour. To find his position relative to the town after 5 hours.

In 1 hour his distance to the east of **A** is $9-3$ miles, or 6 miles;
in 2 hours ,, ,, ,, $6-3$ miles, or 3 miles.

Fig. 22.

Now, in both cases, we have subtracted 3 from the number of miles east of **A** for each hour that he has walked. *Suppose now that we continue repeating the same process;* we shall be led to the result that

in 3 hours his distance to the east of **A** is $3-3$ miles, or 0 miles,
in 4 hours ,, ,, ,, $0-3$ miles, or -3 miles,
in 5 hours ,, ,, ,, $-3-3$ miles, or -6 miles.

But it is evident that after 3 hours' walking the man arrives at **A**, and, if he continues to walk westwards for 2 more hours, he will have then arrived at a place **C**, 6 miles to the west of **A**.

Hence we conclude that -6 miles east of **A** is to be interpreted as signifying 6 miles west of **A**.

By reasoning such as the above, we are led to infer that,

if a distance measured in one direction be represented by a, then $-a$ may be properly and conveniently interpreted as representing an equal distance measured in the opposite direction.

If **OA** (measured from **O**) represent the length a, then $-a$ will be represented by **OA′**, where **A′** is a point on **AO** produced, such that **A′O** = **OA**.

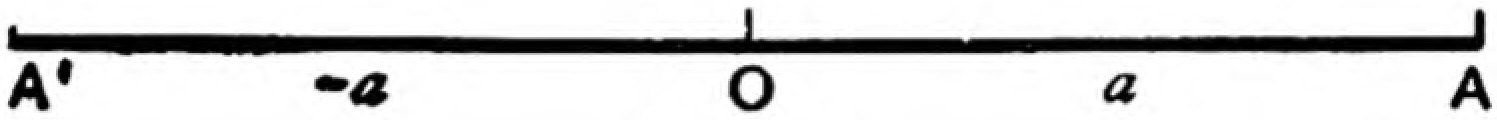


Fig. 23.

30. **Importance of the order of letters.**—The direction in which a line is measured is represented by the order of the letters used in naming the line. Thus **AB** represents a line when it is supposed to be measured from **A** to **B**; but, if the same line be measured from **B** to **A**, we shall call it **BA**, and not **AB**.

If, for example, **A**, **B** are two places, say 12 miles apart, **AB** will denote the distance traversed by a man who walks from **A** to **B**, and **BA** the distance traversed by a man who walks from **B** to **A**. If the former be represented by $+12$, the latter will be represented by -12. If the man walks from **A** to **B** and back again to **A**, his distance from **A** will then be zero, and this fact is represented by the statement

Fig. 23a.

$$\mathbf{AB}+\mathbf{BA} = 12+(-12) = 0.$$

31. **Application of Signs to Trigonometry.**—In defining the trigonometric functions of the angle described by a revolving line **OP** which starts from the position **OX**, the following rules are observed:—

Lines measured along or parallel to **OX** are considered positive when they are drawn in the direction **OX**, negative when they are drawn in the opposite direction **OX′**.

If **OY** be drawn perpendicular to **OX** and *bounding the first quadrant*, lines perpendicular to **OX** are considered positive when they are drawn in the direction **OY**, negative when they are drawn in the opposite direction **OY′**.

In drawing a figure, ***the student should invariably take the***

line **OX** *horizontal and pointing to the right.** **OY** will then be vertical and upwards, and the rules should be remembered in the following form†:—

Horizontal distances to the **right** *of* **OY** *are* **positive** $\}$ .... **(12)**
,, ,, **left** ,, **negative**

Vertical distances **above OX** *are* **positive** $\}$ .... **(13)**
,, ,, **below** ,, **negative**

Thus, in Fig. 24, the horizontal lines **OM, NP, N′S** are all positive, while **OM′, NQ, N′R** are all negative.

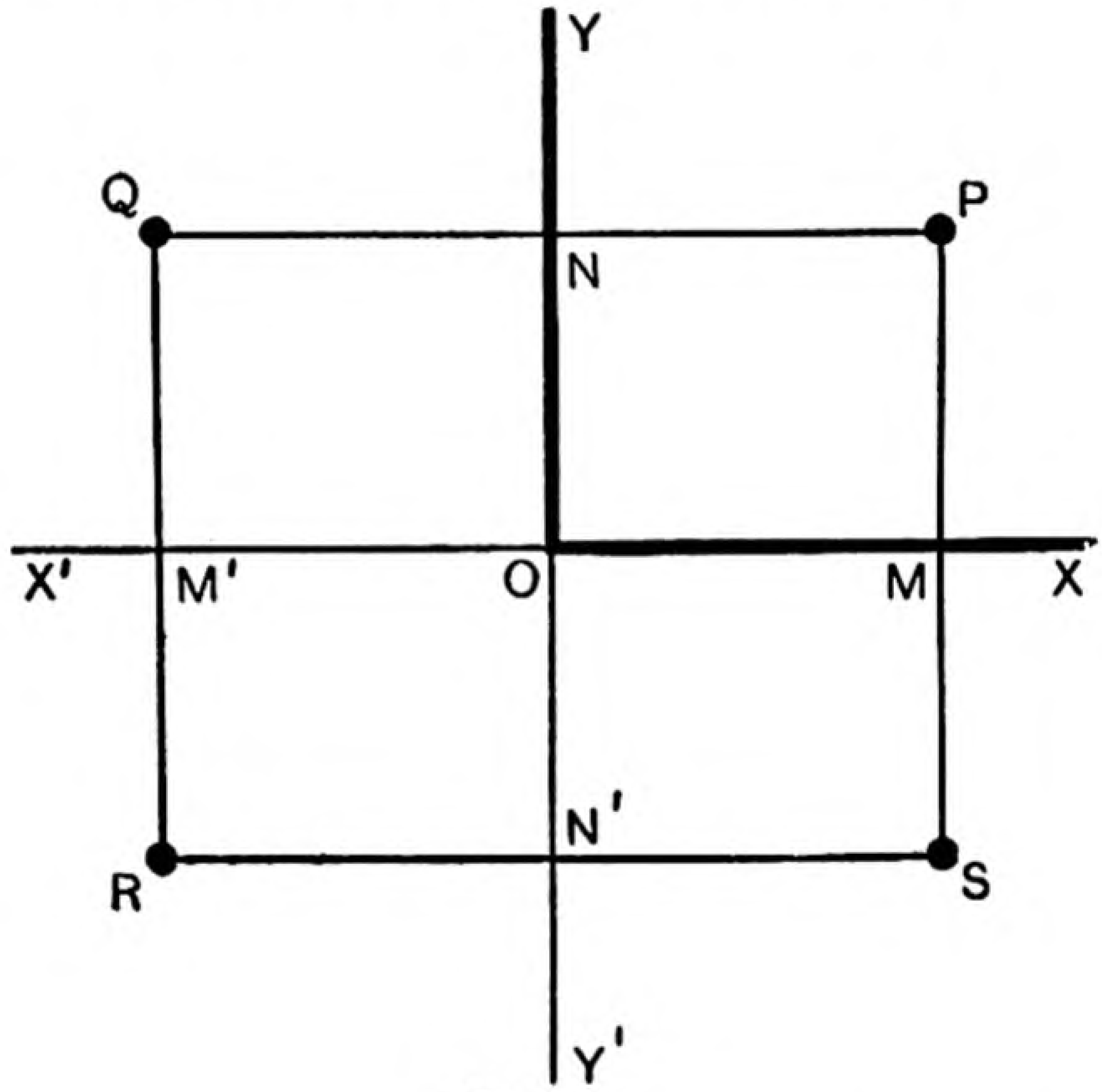


Fig. 24.

So, too, the vertical lines **ON, MP, M′Q** are all positive, while **ON′, MS, M′R** are all negative.

Before defining the trigonometrical functions generally, let us consider functions of an obtuse angle.

---

* If the initial position of the revolving line be in any other direction, *turn the figure round* till its direction is horizontal and points to the right, then indicate the positive directions by the rule.

† The sign to be given to a distance may be thus remembered:—
"Plus to the right; Minus to left;
Positive, height; Negative, depth."

Let the line **OP**, starting from the position OP_1 along **OA**, revolve about **O** through 180°, occupying successively the positions OP_2, OP_3, OP_4, OP_5. During this change of position of **OP**, the perpendicular **MP** from **OA** remains positive, while **OM**, considered positive as long as **M** lies to the right of **O**, or the angle **AOP** is acute, becomes negative when, by the increase of this angle beyond 90°, **M** passes to the left of **O**. Therefore the sine and cosine of an acute angle are each positive, while for angles between 90° and 180° the sine is positive and the cosine is negative.

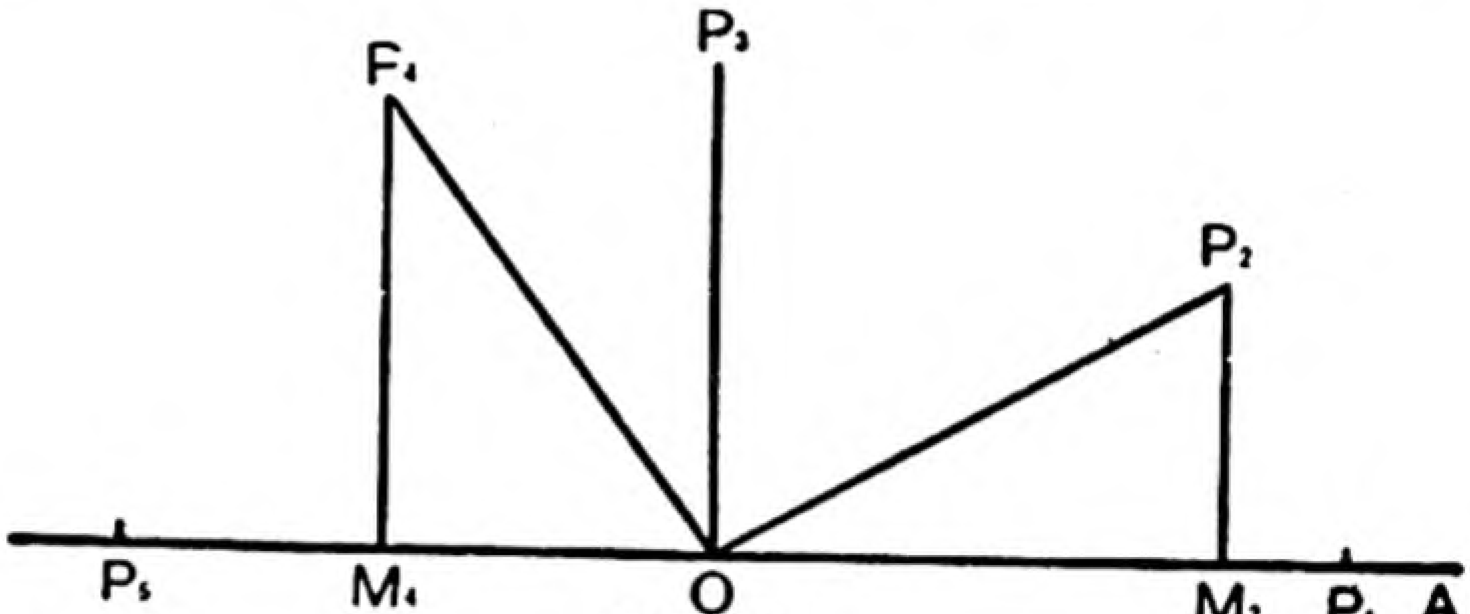


Fig. 25.

32. **Positive and negative angles.**—In trigonometry, angles are regarded as **positive** when they are described by a line or radius vector revolving in the opposite direction to that in which the hands of a clock turn, and this direction is called **counter-clockwise.**

When the radius vector revolves in the same direction as the hands of a clock, or, as it is called, **clockwise**, the angles which it describes are **negative.***

Thus, if the radius vector revolves from **OX** to **OY**, it describes an angle of $+90°$; but, if it revolves in the other direction from **OX** to **OY′**, it describes an angle of $-90°$ (Fig. 24).

We are now in a position to define the trigonometric functions of an angle without any limitation as to the size of the angle.

33. TRIGONOMETRIC FUNCTIONS OF ANY ANGLE.

DEF.—Let **OA** be fixed, and let the revolving line **OP**, starting from **OA**, describe any angle about **O**. Draw **PM**

* "As the hands of a clock go round on the face,
Four negative right angles each time they trace."

always perpendicular to **OA** or **OA′**. Then, giving **MP, OM** their proper signs according to § 31, *and taking* **OP** *always positive*,* the ratio

$\frac{\mathsf{MP}}{\mathsf{OP}}$ is the **sine**, $\frac{\mathsf{OP}}{\mathsf{MP}}$ is the **cosecant**,

$\frac{\mathsf{OM}}{\mathsf{OP}}$ is the **cosine**, $\frac{\mathsf{OP}}{\mathsf{OM}}$ is the **secant**,

$\frac{\mathsf{MP}}{\mathsf{OM}}$ is the **tangent**, $\frac{\mathsf{OM}}{\mathsf{MP}}$ is the **cotangent**,

of the angle AOP.

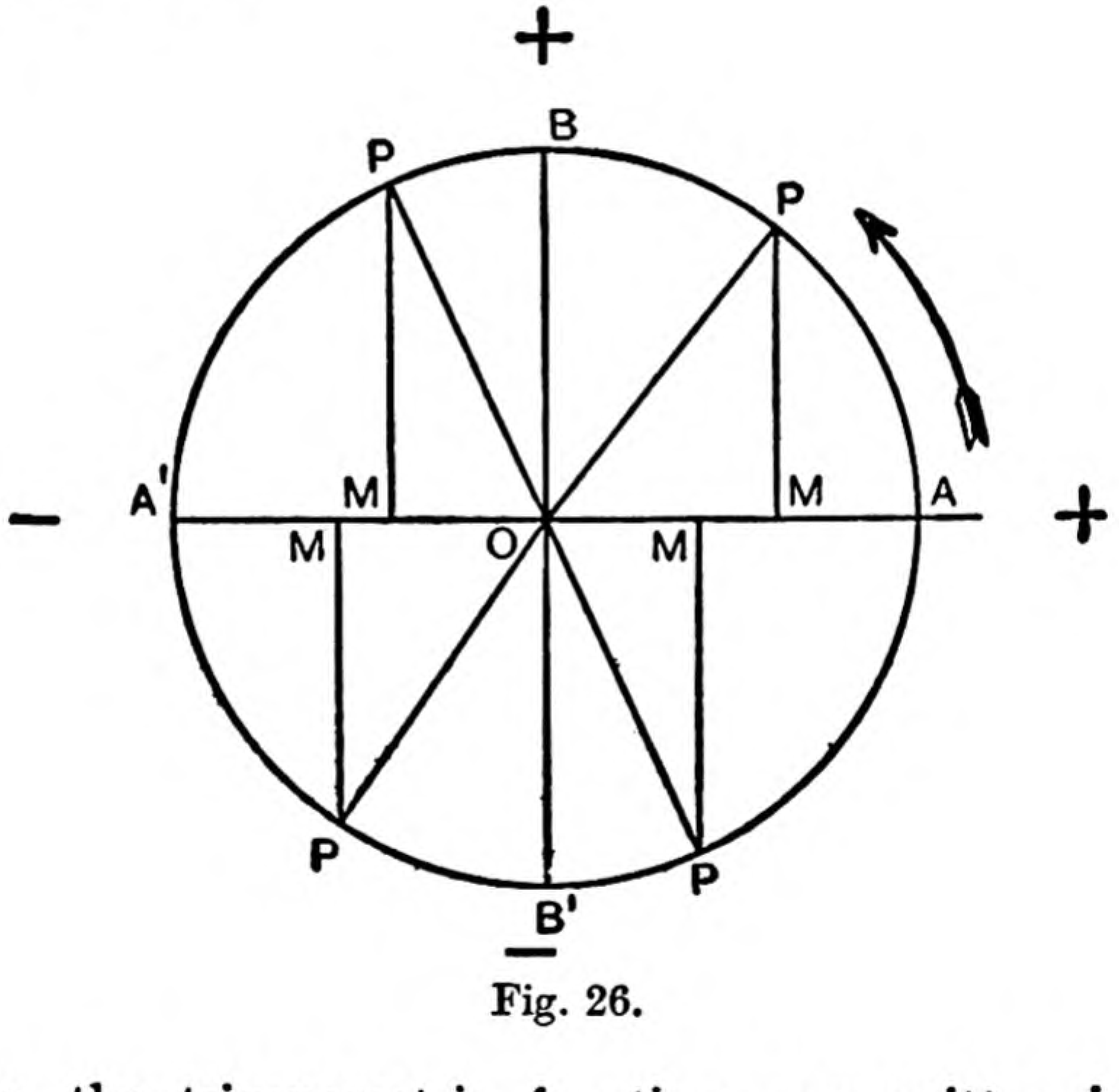


Fig. 26.

When the trigonometric functions are written in their usual abbreviated forms, these definitions stand thus:

* *The radius vector* **OP** *which bounds the given angle is always positive* whatever be its direction. As in Coordinate Geometry, **OP** *would* be negative if the line bounding the angle were produced through **O**, and **P** were taken on the produced part; but such cases hardly ever occur in Trigonometry.

$$\left.\begin{aligned}\sin \mathrm{AOP} &= \frac{\mathrm{MP}}{\mathrm{OP}}, & \operatorname{cosec} \mathrm{AOP} &= \frac{\mathrm{OP}}{\mathrm{MP}}\\ \cos \mathrm{AOP} &= \frac{\mathrm{OM}}{\mathrm{OP}}, & \sec \mathrm{AOP} &= \frac{\mathrm{OP}}{\mathrm{OM}}\\ \tan \mathrm{AOP} &= \frac{\mathrm{MP}}{\mathrm{OM}}, & \cot \mathrm{AOP} &= \frac{\mathrm{OM}}{\mathrm{MP}}\end{aligned}\right\} \quad \cdots\cdots\cdots (14)$$

34. The triangle **OMP** is in every case the **triangle of reference** or **auxiliary triangle** for the angle **AOP**, and, as in § 23, **OM**, **MP**, **PO** *may* be called the base, perpendicular, and hypotenuse if this be a help to remembering the definitions, the functions not defined in the last chapter being

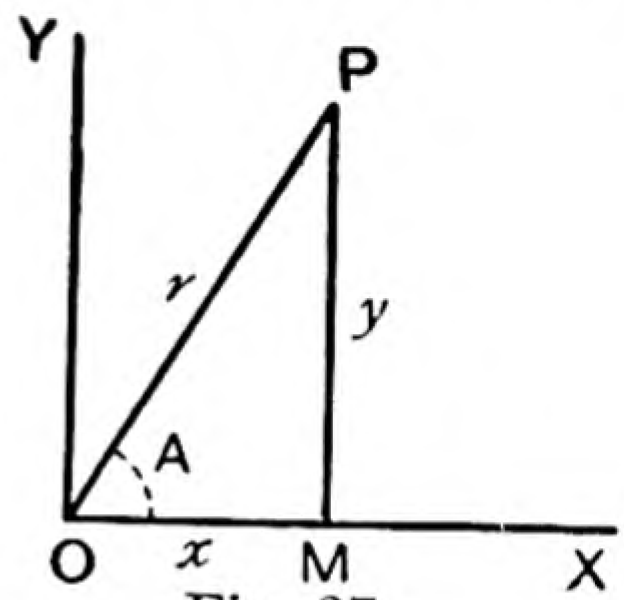


Fig. 27.

$$\operatorname{cosec} = \text{hyp./perp.},\quad \sec = \text{hyp./base},$$
$$\cot = \text{base/perp.}$$

But these terms should never be employed in writing out definitions.

A better plan is to follow the language of Coordinate Geometry, and call **MP** the ordinate, **OM** the abscissa, and **OP** the radius vector of a point **P** on the line bounding the given angle. The definitions now run thus:—

$$\left.\begin{aligned}\text{sine} &= \frac{\text{ordinate}}{\text{radius}}, & \text{cosecant} &= \frac{\text{radius}}{\text{ordinate}}\\ \text{cosine} &= \frac{\text{abscissa}}{\text{radius}}, & \text{secant} &= \frac{\text{radius}}{\text{abscissa}}\\ \text{tangent} &= \frac{\text{ordinate}}{\text{abscissa}}, & \text{cotangent} &= \frac{\text{abscissa}}{\text{ordinate}}\end{aligned}\right\} \quad \cdots\cdots\cdots (15)$$

35. The cotangent of an angle is the reciprocal of the tangent.

Let $$\angle \mathrm{XOP} = A.$$

Then $$\cot A \,.\, \tan A = \frac{\mathrm{OM}}{\mathrm{MP}} \cdot \frac{\mathrm{MP}}{\mathrm{OM}} = 1;$$

$$\therefore\quad \cot A = \frac{1}{\tan A} \quad \cdots\cdots\cdots\cdots (16)$$

This property (like those next to be proved) is *algebraically* true, whether **OM**, **MP** are positive or negative.

36. The secant of an angle is the reciprocal of the cosine, and the cosecant is the reciprocal of the sine.*

For $$\sec A \cdot \cos A = \frac{\mathsf{OP}}{\mathsf{OM}} \cdot \frac{\mathsf{OM}}{\mathsf{OP}} = 1;$$

$$\therefore \quad \sec A = \frac{1}{\cos A} \qquad \text{(17)}$$

and $$\operatorname{cosec} A \cdot \sin A = \frac{\mathsf{OP}}{\mathsf{MP}} \cdot \frac{\mathsf{MP}}{\mathsf{OP}} = 1;$$

$$\therefore \quad \operatorname{cosec} A = \frac{1}{\sin A} \qquad \text{(18)}$$

Since the cosecant, secant, and cotangent are the reciprocals of the sine, cosine, and tangent, respectively, the principal properties of the three former ratios can be readily deduced from those of the latter, and they need not therefore be considered in such great detail.

37. **Versed and coversed sines:**—

$1 - \cos A$ is called the **versed sine of A**, and written **vers A**,

$1 - \sin A$ is called the **coversed sine of A**, and written **covers A**.

If $\mathsf{OA} = \mathsf{OP}$ in Fig. 26, then

$$\text{vers } A = \mathsf{MA}/\mathsf{OP}.$$

We shall sometimes find it convenient to refer to the sine, cosine, and tangent as the three *principal* trigonometric functions of an angle.

38. Signs of the trigonometric functions.—The signs of the lines **OM**, **MP** (Fig. 26) follow the rules of § 31, while **OP** is always positive. Hence

sin AOP is positive if **P** be above **AA′**, *i.e.* for angles in the **first and second quadrants**;

sin AOP is negative if **P** be below **AA′**, *i.e.* for angles in the **third and fourth quadrants**;

cos AOP is positive if **P** be to the right of **BB′**, *i.e.* for angles in the **first and fourth quadrants**;

cos AOP is negative if **P** be to the left of **BB′**, *i.e.* for angles in the **second and third quadrants**;

* The student might more naturally have expected the secant to be the reciprocal of the sine and the *co*secant of the *co*sine, but this is not so. It may be noticed that secant × *co*sine = *co*secant × sine (each being = 1).

tan AOP is positive if **OM**, **MP** be both positive or both negative,* *i.e.* for angles in the **first and third quadrants**;

tan AOP is negative if **OM**, **MP** be one positive and one negative, *i.e.* for angles in the **second and fourth quadrants.**

The cosecant, secant, and cotangent of any angle have the same sign as their reciprocals, viz. the sine, cosine, and tangent, respectively.

Hence the signs of the six trigonometric functions in the different quadrants may be conveniently indicated as in the diagrams below.

SINE AND COSECANT.

+	+
2nd Q.	1st Q.
−	−
3rd Q.	4th Q.

COSINE AND SECANT.

−	+
2nd Q.	1st Q.
−	+
3rd Q.	4th Q.

TANGENT AND CONTAGENT.

−	+
2nd Q.	1st Q.
+	−
3rd Q.	4th Q.

Ex. What are the signs of the functions of 225°?

225° lies between $2\times 90°$ and $3\times 90°$, *i.e.* between 2 and 3 right angles. It is therefore in the third quadrant, and has its sin and cosec *negative*, cos and sec *negative*, tan and cot *positive*.

CAUTION 1.—When the cosine of an angle is negative, it is still represented by $\frac{\mathbf{OM}}{\mathbf{OP}}$, and not $-\frac{\mathbf{OM}}{\mathbf{OP}}$, for **OM** *is itself a minus quantity.*

Thus, if $\mathbf{OP} = 5$ in. and **M** is 4 in. to the left of **O**, then

$$\mathbf{OM} = -4, \text{ and } \cos \mathbf{AOP} = \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{-4}{5} = -\tfrac{4}{5}.$$

The same is true in the case of the other functions.

CAUTION 2.—In naming the trigonometric functions, *care must be taken to write the letters in the right order;* and **not** (as is sometimes done, even in textbooks) to write sin **AOP** as **PM/OP**, but as **MP/OP**. For **PM** represents a length of opposite sign to **MP**, as explained in § 31.

* For a fraction is positive if its numerator and denominator are of like sign, *i.e.* both positive or both negative, and it is negative if they are of unlike sign.

39. **Summary.**—The principal functions which are *positive* in the first, second, third, and fourth quadrants respectively are

all, sin, tan, cos ...... (19)

The reciprocals of these functions, viz.

all, cosec, cot, sec ...... (19A)

are of course also positive in the same quadrants. All the other functions not mentioned as positive in any quadrant are negative.

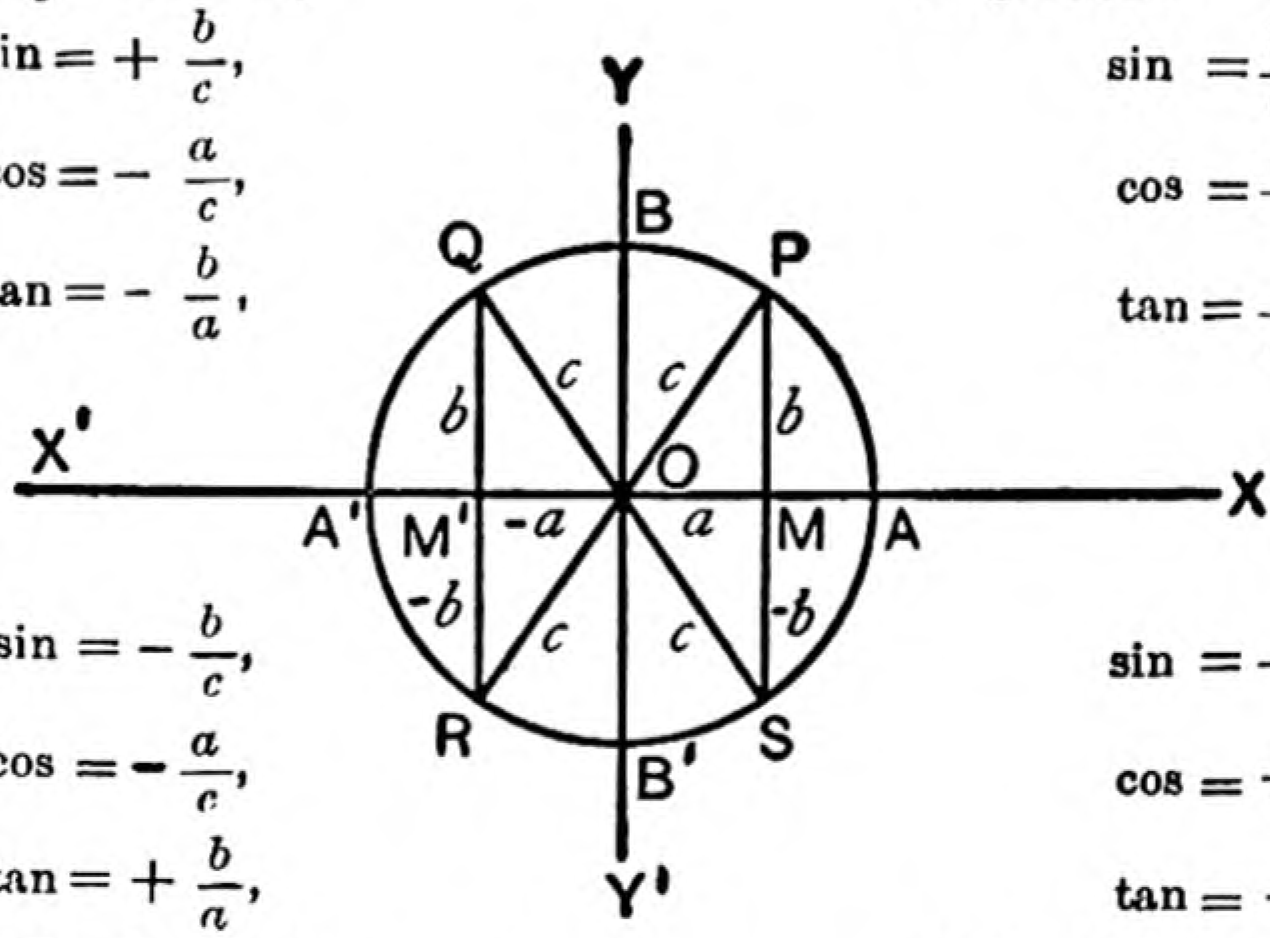
40. The definitions of the trigonometric functions and their signs in different quadrants may also be illustrated in a slightly different way.

Let **AOP** be an angle A in the first quadrant.

Let a, b, c be the *lengths* (without regard to sign) of the "base," "perpendicular," and "hypotenuse" of its triangle of reference; then, by measuring off lengths equal and opposite to the two former, we obtain the triangles of reference of angles **AOP, AOQ, AOR, AOS,** in the four quadrants, respectively, and it will easily be seen that the principal trigonometrical functions of these angles are as indicated in Fig. 28.*

Angle **AOQ** $=180^\circ - A$,
$\sin = +\dfrac{b}{c}$, $\cos = -\dfrac{a}{c}$, $\tan = -\dfrac{b}{a}$,

Angle **AOP** $= A$,
$\sin = \dfrac{b}{c}$, $\cos = \dfrac{a}{c}$, $\tan = \dfrac{b}{a}$,



$\sin = -\dfrac{b}{c}$, $\cos = -\dfrac{a}{c}$, $\tan = +\dfrac{b}{a}$,
Angle **AOR** $=180^\circ + A$.

$\sin = -\dfrac{b}{c}$, $\cos = +\dfrac{a}{c}$, $\tan = -\dfrac{b}{a}$,
Angle **AOS** $=360^\circ - A$.

Fig. 28.

For example, by definition, tan **AOR** = **M′R/OM′**,

but **M′R** $=-b$, **OM′** $=-a$; $\therefore$ tan **AOR** $= \dfrac{-b}{-a} = \dfrac{b}{a}$.

---

* The figure also serves to connect the trigonometrical functions of the four angles, A, $180^\circ - A$, $180^\circ + A$, $360^\circ - A$. The relations between these will be fully discussed in Chapter VIII.

ILLUSTRATIVE EXERCISE.

Write down the cosecant, secant, and cotangent of each of the angles in Fig. 28.

41. **Limits to the values of the trigonometric functions.**— Whatever be the magnitude of the angle considered, the fundamental triangle **OMP** is right-angled at **M**, and its interior *acute* angles **MOP, OPM** are therefore neither of them greater than $\angle$**OMP.** Hence, since the greater angle is subtended by the greater side,

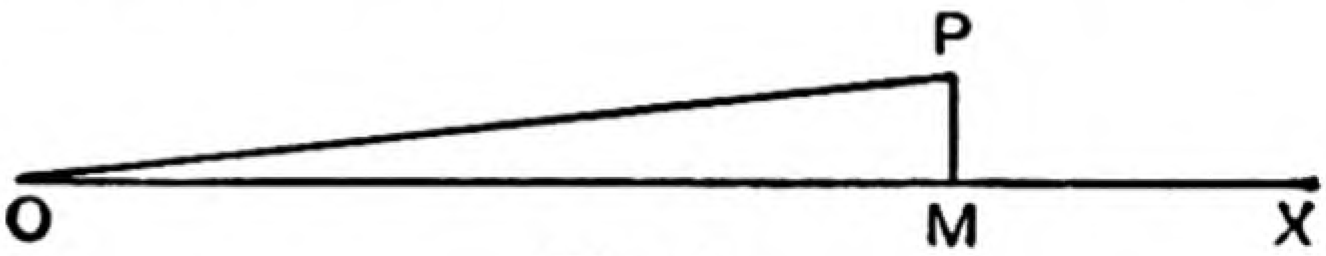


Fig. 29.

$\therefore$ **MP** and **OM** are never numerically $>$ **OP**;

$$\therefore \quad \frac{\mathbf{MP}}{\mathbf{OP}} \text{ and } \frac{\mathbf{OM}}{\mathbf{OP}} \text{ are never numerically } > 1.$$

Hence the sine and cosine of an angle are never numerically greater than unity, that is, they never lie beyond the limits $+1$ and -1.

Their values are therefore always *proper fractions.*

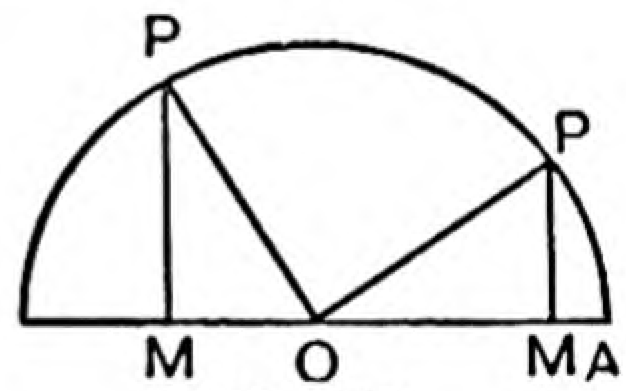


Fig. 30.

Conversely, **OP** is never numerically $<$ **MP** or **OM**;

$$\therefore \quad \frac{\mathbf{OP}}{\mathbf{MP}} \text{ and } \frac{\mathbf{OP}}{\mathbf{OM}} \text{ are never numerically } < 1.$$

Hence the cosecant and secant of an angle are never numerically less than unity, that is, they never lie between the limits $+1$ and -1.

On the other hand, by making the acute $\angle$**OPM** small enough we can make **OM** as small a fraction of **MP** as we please, and therefore can make **MP** as large a multiple of **OM** as we please.

Therefore **MP/OM** may be made larger, and **OM/MP** smaller, than any assigned numerical quantity.

Similarly, making $\angle$ **MOP** small enough, **MP**/**OM** may be made smaller, and **OM**/**MP** larger, than any assigned numerical quantity.

Hence there is no limit to the values of the tangent and cotangent of an angle, and they may have any values whatever, positive or negative.

Summary:—

$$\left.\begin{array}{lll} \text{sin and cos} & = \text{or} < 1 & \text{(numerically)} \\ \text{sec and cosec} & = \text{or} > 1 & \quad ,, \\ \text{tan and cot} > & = \text{or} < 1 & \quad ,, \end{array}\right\} \ldots\ldots (20)$$

42. Given the sine or cosecant of an angle, to construct the angle.

Let the sine be given $= p/r$, a positive or negative fraction whose denominator r is positive.

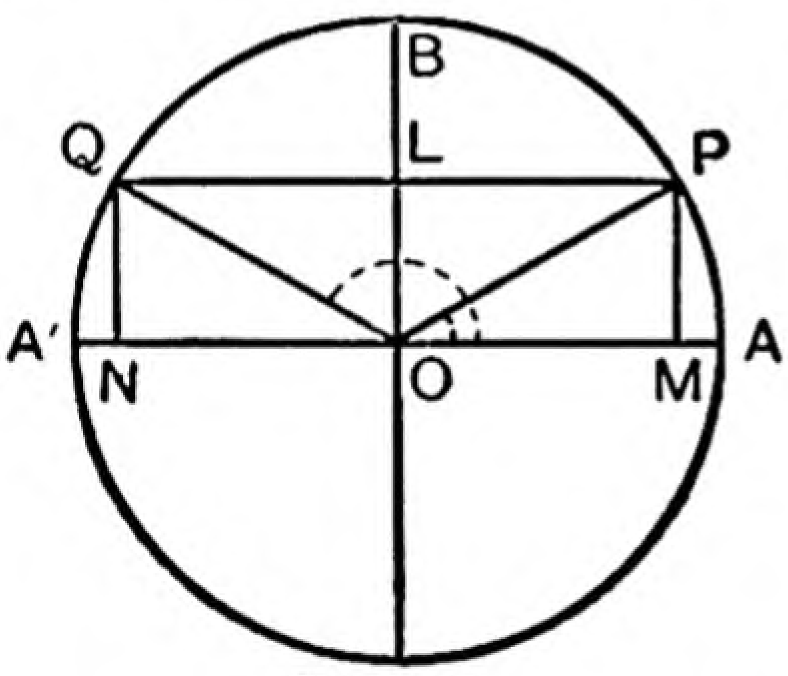


Fig. 31.

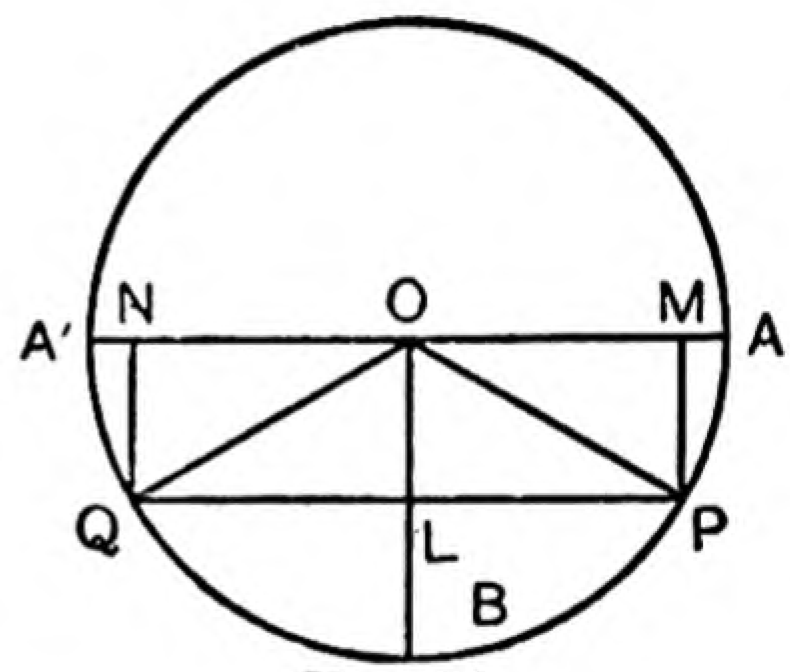


Fig. 32.

Take **OA** the primitive line.

About **O** as centre, and with radius $= r$, describe a circle. Draw **OB** at right angles to **OA**, and on **OB** measure off **OL** equal to p, and of the same algebraic sign as p. Draw **PLQ** through **L** parallel to **AO**, cutting the circle in **P** and **Q**. Then $\angle$ **AOP** and $\angle$ **AOQ** are angles having the given sine.

For, if **PM**, **QN** be drawn perpendicular on **OA**, then, by construction

$$\sin \mathbf{AOP} = \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{p}{r} \text{ and } \sin \mathbf{AOQ} = \frac{\mathbf{NQ}}{\mathbf{OQ}} = \frac{p}{r},$$

Either of the angles **AOP** or **AOQ** will be a solution of the problem.

If p/r is *negative*, **L** must be taken *below* **O** as in Fig. 32.

Again, if the *cosecant* be given $= r/p$, the sine will be $= p/r$, and the construction will be the same as before.

If $p > r$, the line through **L** will not cut the circle; this accords with the fact that the sine of an angle cannot be numerically > 1.

43. Given the cosine or secant of an angle, to construct the angle.

Let the cosine be given $= q/r$, a positive or negative fraction, r being positive.

Take **OA** as primitive line.

About **O** as centre and radius $= r$, describe a circle. On **OA** cut off **OM** algebraically $= q$. Draw **PMQ** through **M** at right angles to **AO**, cutting the circle in **Q** and **P**. Then $\angle$**AOP** and $\angle$**AOQ** are angles having the given cosine.

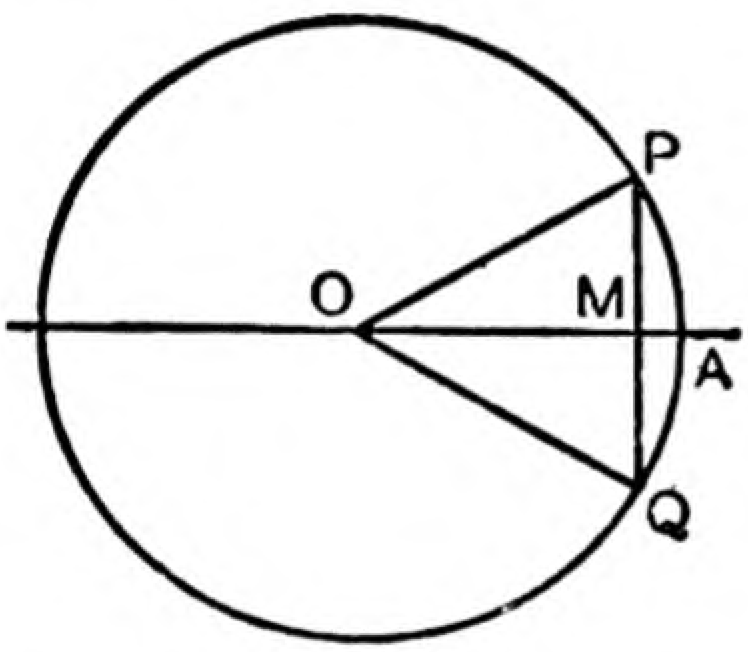


Fig. 33.

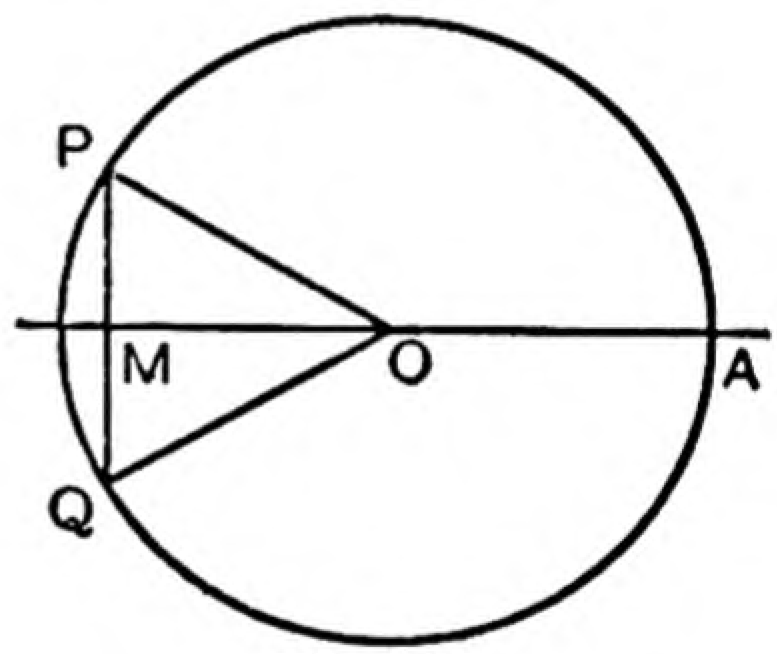


Fig. 34.

For

$$\cos \mathbf{AOP} = \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{q}{r} \text{ and } \cos \mathbf{AOQ} = \frac{\mathbf{OM}}{\mathbf{OQ}} = \frac{q}{r}.$$

Either of the angles **AOP**, **AOQ** will be a solution of the problem.

If q/r is *negative*, **M** must be taken on **AO** produced, as in Fig. 34.

Again, if the *secant* be given $= r/q$, the cosine will be $= q/r$, and the construction will be the same as before.

The construction, like that of the preceding article, fails if $q > r$, or q/r the given cosine > 1.

44. Given the tangent or cotangent of an angle, to construct the angle.

Let the tangent be given $= p/q$, a positive or negative fraction of any magnitude whatever.

Along the primitive line measure off **OM** $= q$. Through **M** draw a line perpendicular to **OM**, and on it cut off **MP** algebraically $= p$. Produce **PO** to **R**. Then $\angle$**AOP** and $\angle$**AOR** are angles having the given tangent.

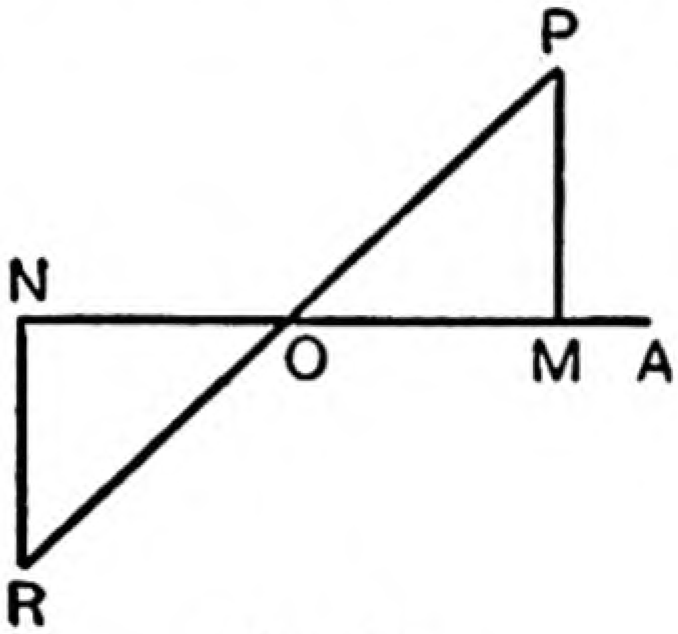


Fig. 35.

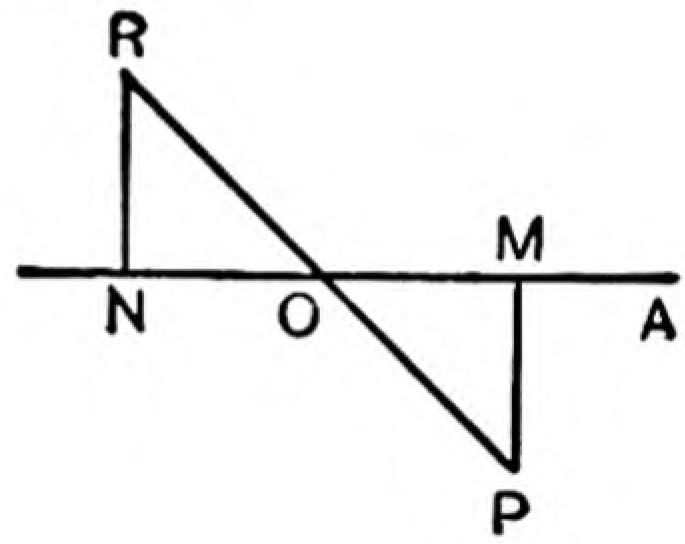


Fig. 36.

For, if **OR** be taken $=$ **OP**, and the triangle **ORN** completed, then

$$\tan \mathbf{AOP} = \frac{\mathbf{MP}}{\mathbf{OM}} = \frac{p}{q} \text{ and } \tan \mathbf{AOR} = \frac{\mathbf{NR}}{\mathbf{ON}} = \frac{-\mathbf{MP}}{-\mathbf{OM}} = \frac{p}{q},$$

Either of the angles **AOP, AOR** will be a solution of the problem.

If p/q is *negative* and **OM** measured to the right, **P** must be taken below **M**, as in Fig. 36.

Again, if the *cotangent* be given $= q/p$, the tangent will be $= p/q$, and the construction will be the same as before.

There is no limitation to the value of p/q in the present construction.

45. **On Projections.—The projection of a given point on a given straight line is the foot of the perpendicular drawn from the point to the line.** Thus in Fig. 37 the projection of the point **B** on the line **XY** is the point *b*.

The projection of a line on a given straight line is that portion of the given straight line which lies between the projections of the two extremities of the given line. Thus in

Fig. 37 the projection of the line **AB** on the straight line **XY** is the line *ab*.

The first of the two lines (the line which is projected) may be either *straight*, or *curved*, or "*broken*" (*i.e.* consisting of a series of straight or curved lines joined together). The second of the two lines (on to which the first is projected) *must however be a straight line*, since the definition involves perpendiculars to this line. Thus in Fig. 37 the projection of the curved line **CD** on the straight line **XY** is *cd*; and in Fig. 38 the projection of the broken line **EFGHK** on the straight line **PQ** is *ek*.

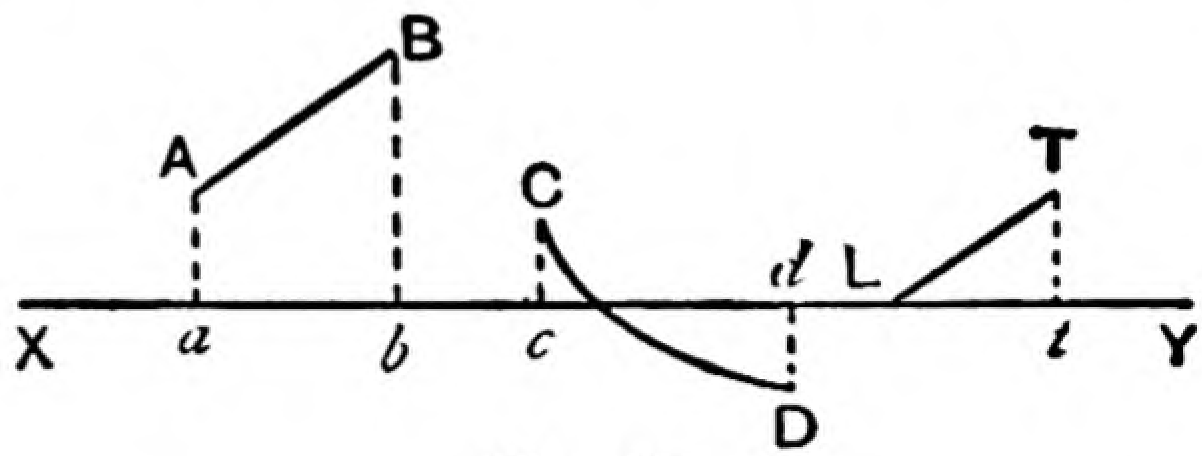


Fig. 37.

Note that in Fig. 37 the projection of the point **L** is **L** itself, and the projection of the line **LT** is **L***t*.

In projections we generally adopt the usual convention of signs, viz. that when the letters of the projection are quoted *in one order* (usually from left to right) the projection is reckoned *positive*, and when quoted in the reverse order the projection is reckoned negative. Thus in Fig. 37 the projection of the line **CD** is *cd* which is reckoned positive, while the projection of the line **DC** is *dc* which is reckoned negative.

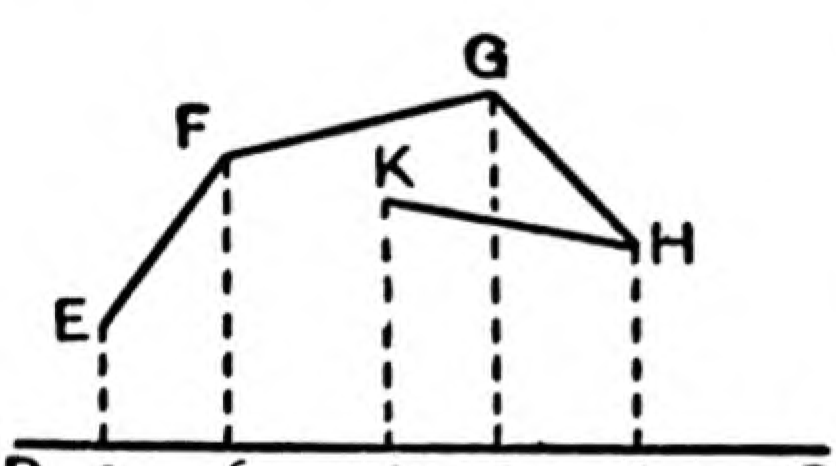


Fig. 38.

Adopting this convention it is easy to see that **the projection of a broken line is equal to the algebraic sum of the projections of its component parts.** For example, in Fig. 38, the projection of the broken line **EFGHK** is *ek* (by definition). Also the algebraic sum of the projections of its component parts **EF, FG, GH, HK** will be $ef + fg + gh + hk$,

i.e. $ef + fg + gh - kh$,

which is again equal to *ek*.

To find the algebraic projection of one straight line on another.

In Fig. 39 it is required to find the projection of **OH** on **PQ**.

Draw **OX** *parallel to the positive direction of* **PQ**.

Then projection of

$$\mathbf{OH} = oh = \mathbf{OM} = \mathbf{OH}.\frac{\mathbf{OM}}{\mathbf{OH}} = \mathbf{OH}.\cos \mathbf{HOX}.$$

Thus to find the projection of OH we multiply it by the cosine of the angle which OH makes with OX, *i.e.* with the positive direction of **PQ**.

The angle **HOX** is positive and acute, but it will be found that the above rule holds, whatever the inclination of the projected line may be.

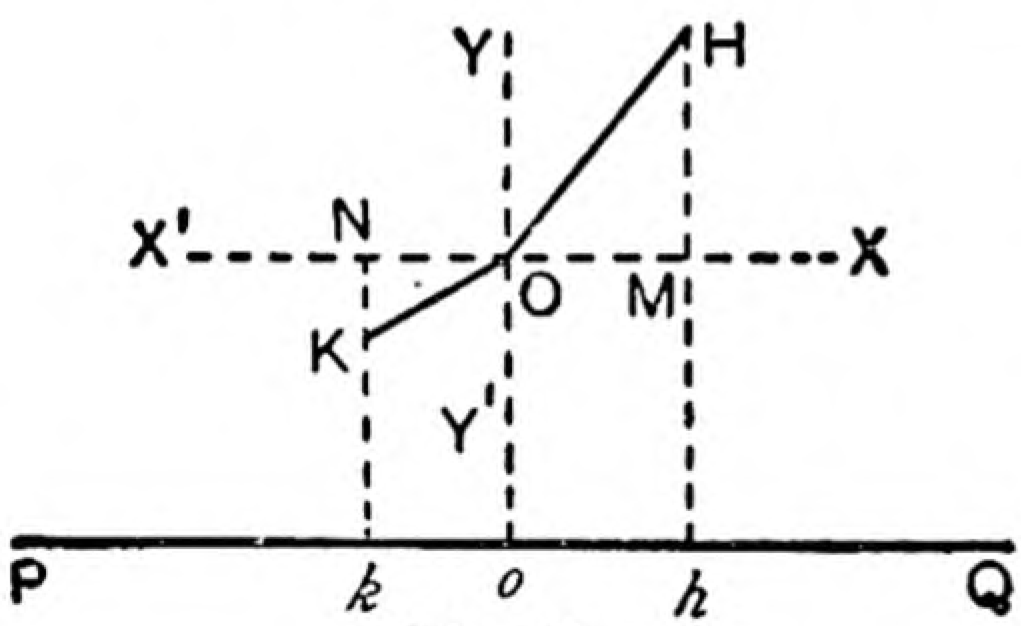


Fig. 39.

Thus if it be required in Fig. 39 to find the projection of the line **OK**:—

In whatever position **OK** may fall, we may use the triangle **KON** as the triangle of reference to determine the ratios of the angle **KOX**. Thus $\cos \mathbf{KOX} = \frac{\mathbf{ON}}{\mathbf{OK}}$.

Hence, adopting throughout the usual conventions of sign, we have for all positions of **OK**

$$\text{projection of } \mathbf{OK} = ok = \mathbf{ON} = \mathbf{OK}.\frac{\mathbf{ON}}{\mathbf{OK}}$$

$$= \mathbf{OK}.\cos \mathbf{KOX}.$$

Generally, therefore, if a line containing l units of length be projected *on to another line,* **with which it makes an angle a,**

$$\text{length of projection} = l \cos a.$$

Similarly, if the line of length l be projected on to another line perpendicular to the first line of projection or making an angle $\left(\frac{\pi}{2} - a\right)$ with the line of length l, then

$$\text{length of projection} = l \cos\left(\frac{\pi}{2} - a\right) = l \sin a.$$

For example, in Fig. 39, the projection of **OH** on any line perpendicular to **PQ** will be equal to **MH**, *i.e.* to $\mathbf{OH}.\dfrac{\mathbf{MH}}{\mathbf{OH}}$, or to **OH** sin **HOX**. Also the projection of **OK** on any line perpendicular to **PQ** will be equal to **NK**, *i.e.* to $\mathbf{OK}.\dfrac{\mathbf{NK}}{\mathbf{OK}}$, or to **OK**.sin **KOX.**

EXAMPLES IV.

1. If **P** be any point in the same straight line as **A** and **B**, show that for all positions of **P** the equation **AP**+**PB** = **AB** will hold good, provided proper attention is paid to the convention as to positive and negative lengths.

2. If **C** be taken anywhere in the straight line which passes through **A** and **B**, prove that, whatever the position of **C**, as long as **A** and **B** are fixed, the value of **BA**+**BC**+**CA** must be the same.

3. Define the *trigonometrical ratios of an angle*, illustrating their names by reference to a figure.

4. A denotes an angle greater than three, and less than four, right angles. Show in a diagram, the angle, and in another diagram the angle $180°-A$, taking care so to letter the diagram as to leave no doubt as to your meaning.

5. Treat the angle 479° in the same way as the angle A in Question 4.

6. Treat the angle 9847° in the same way.

7. Write down the signs of **AM**/**AP** when the angle A is greater than two, but less than three, right angles, and when A is 732°, the angle M being a right angle.

8. In Fig. 26 of § 33, discuss at length the sign of **MO**/**PO**, as the angle O increases from 0° to 360°.

9. Show that the values of the trigonometrical functions are not altered by altering the position of **P** in the revolving line **OP**.

10. Construct an angle whose sine is $\frac{1}{3}$.

11. Construct an angle less than 360° whose tangent is 1, and show that there are two such angles.

12. Construct an angle whose sine is $\frac{1}{8}$, and find its cosine.

13. Construct an angle of 15°; and find, roughly, by actual measurement, the sine of 15°.

14. Draw the positions of the revolving line when the angle has its cotangent equal to -3.

15. If you change the sign of an angle less than a right angle, which of its trigonometrical functions will also have their signs changed?

16. Which of the trigonometrical ratios are never greater, and which are never less, than unity?

17. Can an angle θ exist such that $20 \sec^2 \theta - 3 \sec \theta$ shall be equal to 9?

18. Can an angle θ exist such that $9 \sin^2 \theta + 3 \sin \theta$ shall be equal to 20?

19. Is the equation $\sec \theta = \dfrac{a^2 + b^2}{2ab}$ possible? If so, why?

20. Is the equation $\cos^2 \theta = \dfrac{(a + b)^2}{4ab}$ possible? If so, when?

21. There are four buildings, A, B, C, D, on an extensive plain. The bearings of D from A are not known: but the bearings of B from A, C from B, D from C are respectively 12 miles E. 15° N., 17 miles N. 30° E., 9 miles N. 5° E. Show that D is approximately 26·8 miles N. of A and 20·4 miles E. of A.

22. A motor car runs as follows: 23 miles E. 20° N., 15 miles N., 31 miles N. 10° W., 14 miles N.W., 12 miles W., 56 miles S. Show that it is then 7·298 miles N. and 5·531 E. of its first position.

CHAPTER V.

TRIGONOMETRIC FUNCTIONS OF A VARIABLE ANGLE.

In the preceding chapter we have defined the trigonometric functions of an angle, and have examined in which quadrants these functions are positive and in which negative. We shall now consider a **variable** angle described by a line which is continuously revolving in the positive direction, and shall investigate the manner in which the trigonometric functions increase or decrease and change sign as the angle continuously increases.

46. **Graphic representation of functions.**—It will be convenient at the outset to explain what is meant by the *graphical method* of representing functions in general, and trigonometric functions in particular.

The reader will find it useful to recall any simple illustrations of the method with which he is familiar. We may instance the recording barometer and thermometer which trace out a curve on a sheet of suitably-ruled paper, forming a record of the fluctuations that took place in any given period; the use of curves to illustrate statistical questions, the graphic representation of variable velocities in mechanics, and so on.

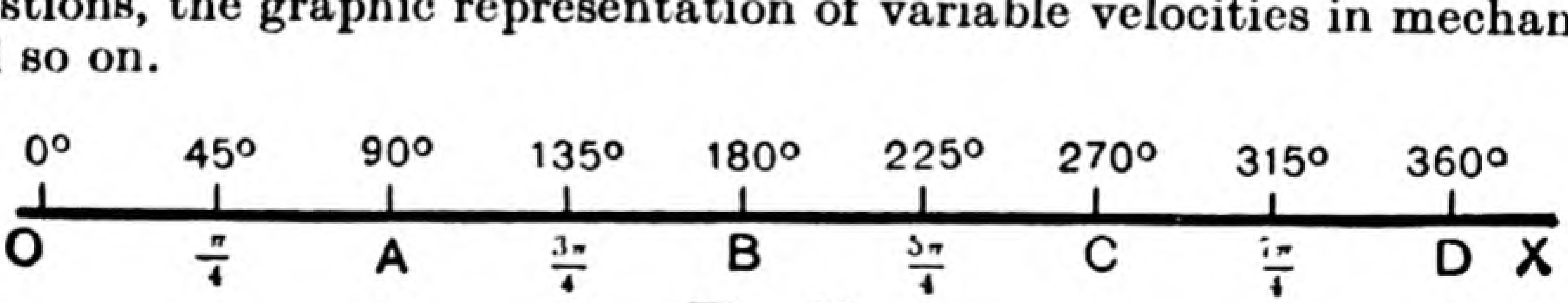


Fig. 40.

Suppose now that it is required to represent the variations of sin θ by means of a curve. The angle θ may be represented in magnitude by means of a straight line by taking this line to contain as many units of length as θ contains units of angular measurement. Taking a radian as the latter unit, let the horizontal line **OD** (Fig. 40) contain 2π (or roughly $6\frac{2}{7}$) units of length; thus **OD** represents an angle of 2π radians, *i.e.* 360°.

Subdividing this line into (say) eight equal parts, the first segment will represent an angle $\frac{1}{4}\pi$ or 45°, **OA** will represent $\frac{1}{2}\pi$ or 90°.

Generally any angle θ will be represented by a length **OM** measured from **O**, containing θ units, and **OM** will be to **OD** as θ to 2π. Now through **M** draw a line **MP** perpendicular to **OD** containing $\sin\theta$ units of length. Then **MP** represents $\sin\theta$. **OM** is called the abscissa, and **MP** the ordinate.

Suppose the lines **OM**, **MP** were drawn for *every* value of the angle θ. Then the points **P** thus obtained would be found to lie on a certain curve, which is sometimes called the *graph** of $\sin\theta$. This curve forms a kind of record of the way in which $\sin\theta$ varies with different values of θ.

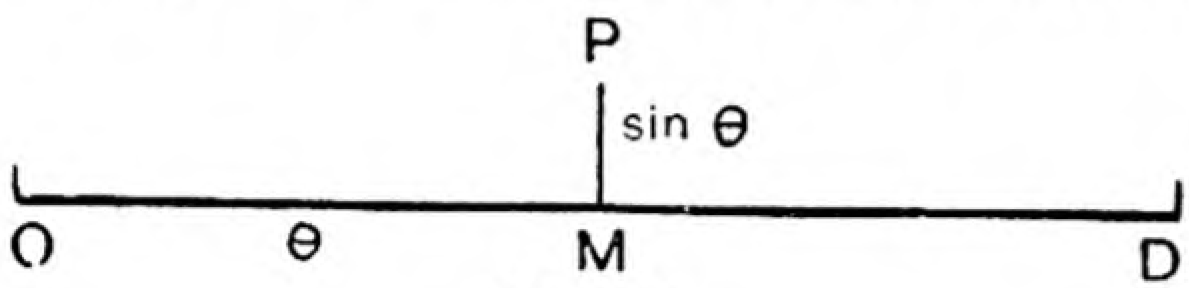


Fig. 41.

Thus, where $\sin\theta$ is positive, the curve is *above* **OX**; where negative, *below*; when $\sin\theta$ increases, the curve rises (going in the direction **O** to **D**); when $\sin\theta$ decreases, it falls; when $\sin\theta = 0$, the curve cuts **OX**; and when $\sin\theta$ is greatest, the curve is at its furthest from **OX**.

The same considerations evidently hold if we represent any other function of θ in a similar way, such as $\cos\theta$, $\tan\theta$, etc.

47. The meaning of infinity.—When a variable quantity can be made larger than any number that can be named or conceived, it is said to become infinite or to *tend to infinity.*

Thus, suppose $y = \dfrac{1}{x}$. If we give x any particular value, we can calculate the corresponding value of y, and by altering the value of x we can alter that of y. Hence y is said to be a variable quantity whose value depends on that of x.

If, now, we make x very small, we can make y very large. For example, if we take $x = \dfrac{1}{100}$, then $y = 100$; if $x = \dfrac{1}{10^6}$, then $y = 10^6$, and so on. In general, if we take any number **N**, however large, we can make y greater than **N** by taking x less than $\dfrac{1}{\mathbf{N}}$. Hence we say that when x approaches zero, y becomes infinite, and we write $y = \infty$. The symbol ∞ thus stands for infinity.

* In the notation of Coordinate Geometry $x = \theta$ and $y = \sin\theta$, so that the equation of the curve is $y = \sin x$.

Caution.—The student should not think of ∞ as a number and say that ∞ is the value of y when $x = 0$. Since division by zero is impossible, $\frac{1}{0}$ is, strictly speaking, meaningless, so that y has no value when $x = 0$. The statement $y = \infty$ when $x = 0$ must always be interpreted as meaning simply that y can be made as large as we please by taking x sufficiently small.

48. The change of sign in passing through infinity.—In the previous paragraph we confined our attention to positive values of x. If we consider negative values of x, we find that when x is very small, y is very large and negative. Thus for negative values of x, y approaches $-\infty$ as x approaches zero.

More generally, if $y = \frac{p}{q}$, where p and q both vary, then as q approaches the value 0, $\frac{1}{q}$, and hence also $\frac{p}{q}$, becomes infinite. If q is positive, y approaches $+\infty$; if q is negative, y approaches $-\infty$. We express this by saying that as q passes from positive values through the value 0 to negative values, $\frac{p}{q}$ changes from $+\infty$ to $-\infty$.

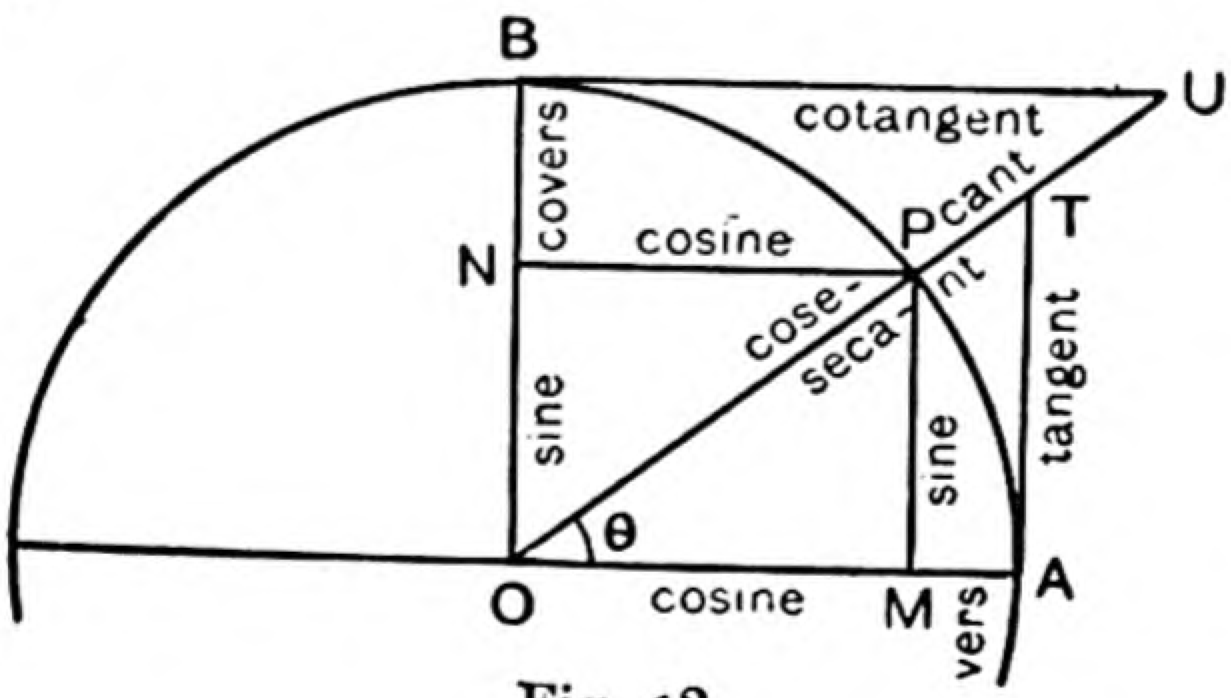


Fig. 42.

49. To represent all the trigonometric functions of an angle on the same scale, by lengths of straight lines, in a single diagram.

Let **AOP** be any angle θ. About **O** describe a circle, and make $\angle$**AOB** $= 90°$. Draw **AT, BU** touching the circle at

A, **B**, and cutting **OP** produced in **T**, **U**. Draw **PM**, **PN** perpendicular on **OA**, **OB**. Then, if r denote the radius of the circle **OP**, it is easy to see that

$$\sin\theta = \frac{\mathsf{ON}}{r} = \frac{\mathsf{MP}}{r}, \qquad \cos\theta = \frac{\mathsf{OM}}{r} = \frac{\mathsf{NP}}{r},$$

$$\tan\theta = \frac{\mathsf{AT}}{r}, \qquad \cot\theta = \frac{\mathsf{BU}}{r},$$

$$\sec\theta = \frac{\mathsf{OT}}{r}, \qquad \operatorname{cosec}\theta = \frac{\mathsf{OU}}{r},$$

$$\operatorname{vers}\theta = \frac{\mathsf{MA}}{r}, \qquad \operatorname{covers}\theta = \frac{\mathsf{NB}}{r}.$$

Hence, if the radius r be taken to be the unit of length, the lengths of the eight lines **MP**, **NP**, **AT**, **BU**, **OT**, **OU**, **MA**, **NB** represent the eight trigonometric functions of the angle **AOP**.

50. **Old definitions of the trigonometric functions.**—According to early writers on Trigonometry, these eight lines were defined as the sine, cosine, tangent, etc., **of the arc AP**, and their magnitudes depended not only on the angle **AOP** of the arc, but were proportional to r, the radius of the circle.* The modern trigonometric functions of an angle are thus simply the corresponding old functions of the arc divided by the radius of the circle.

The arc **PB**, which = one quadrant − arc **AP**, was called the *complement of the arc* **AP**.

Illustrative Exercises.

Prove, from the figure of § 49, that

$$\tan\theta = \sin\theta/\cos\theta, \quad \operatorname{cosec}\theta = 1/\sin\theta, \quad \sec^2\theta = 1+\tan^2\theta,$$

$$\operatorname{cosec}^2\theta = 1+\cot^2\theta, \quad \sin^2\theta = \operatorname{vers}\theta\,(2-\operatorname{vers}\theta), \quad \sin(90^\circ-\theta) = \cos\theta.$$

---

* Produce **PM** to meet the circle again (below **OA**) in **P′**. The figure **PAP′M** bears a fanciful resemblance to a bow and arrow, **M** being the end of the arrow held next the archer's *breast*. Hence the name *sine*, derived from the Latin *sinus*, a bosom. **AT** is the *tangent*, because it touches the circle in **A** (Latin, *tango*, I touch), and **OT**, the *secant*, cuts it (Latin, *seco*, I cut).

51. To trace the variations of the sine of an angle as the angle continuously increases.

Let a line **OP**, of unit length, start from the position **OA** and revolve about **O** continuously in the positive direction, its extremity **P** thus tracing out a circle whose radius = 1.

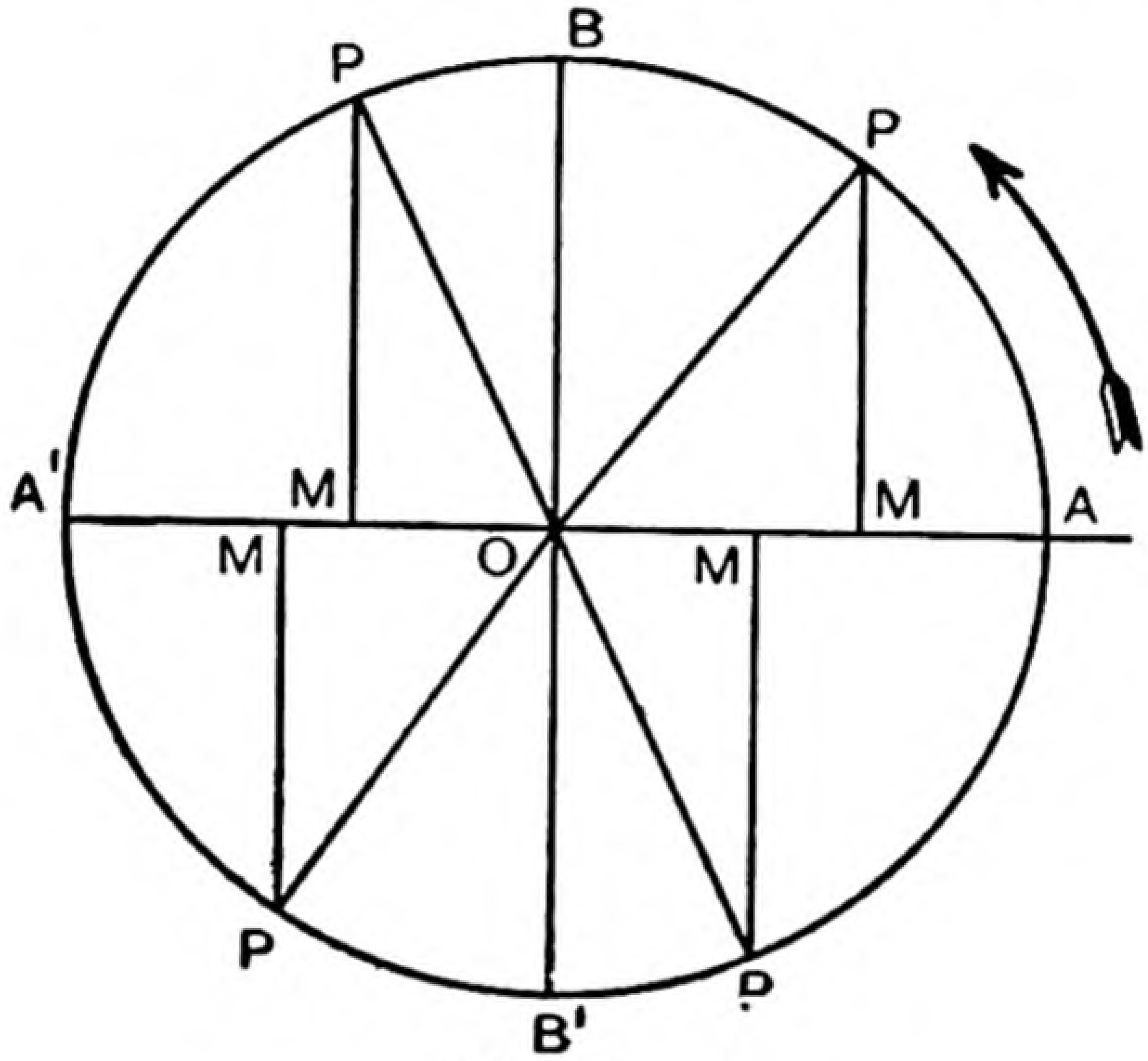


Fig. 43.

In any position, let **PM** be drawn perpendicular on **OA** or **OA** produced.

Then, if θ denote the angle described by **OP**,

$$\sin\theta = \frac{\mathbf{MP}}{\mathbf{OP}} = \mathbf{MP}, \text{ since } \mathbf{OP} \text{ has been taken} = 1.$$

As **P** moves round continuously in the direction of the arrow, **MP** *increases* while **P** moves from **A** to **B**, decreases from **B** to **A′**, increases numerically from **A′** to **B′**, but is negative, and then decreases numerically, remaining negative, from **B′** to **A**. Moreover, the numerically greatest values of **MP** or $\sin\theta$ are **OB** and **OB′**, and are equal to unity.

Now, when a negative quantity increases in numerical magnitude, its algebraic value decreases, for instance, -1 is algebraically less than 0, though its numerical value 1 is greater than 0. Hence we have finally the following results:—

In the first quadrant, as θ increases from 0 to $\frac{1}{2}\pi$,
sin θ increases from 0 to 1, and is positive.

In the second quadrant, as θ increases from $\frac{1}{2}\pi$ to π,
sin θ decreases from 1 to 0, and is positive.

In the third quadrant, as θ increases from π to $1\frac{1}{2}\pi$,
sin θ decreases from 0 to -1, and is negative.

In the fourth quadrant, as θ increases from $1\frac{1}{2}\pi$ to 2π,
sin θ increases from -1 to 0, and is negative.

When **OP** has described 2π, it is again at **OA**, and it subsequently revolves over the same ground again. Hence the changes in sin θ when θ is between 2π and $2\frac{1}{2}\pi$ are the same as when θ is between 0 and $\frac{1}{2}\pi$; its changes between $2\frac{1}{2}\pi$ and 3π are the same as between $\frac{1}{2}\pi$ and π, and so on, and the same cycle of changes repeats itself indefinitely every time θ increases by 2π.

This fact is expressed by the statement that sin θ is a **periodic function of** θ, its **period** being 2π.

52. To trace the variations of the cosine of an angle as the angle continuously increases.

Take the figure and construction of the last article. Then

$$\cos\theta = \frac{\mathsf{OM}}{\mathsf{OP}} = \mathsf{OM}, \text{ since } \mathsf{OP} = 1.$$

As **P** revolves round the circle **ABA′B′** in the direction of the arrow, **OM** decreases from **OA** to zero while **P** moves from **A** to **B**; increases numerically, but is negative, while **P** moves from **B** to **A′**; decreases numerically, but remains negative, while **P** moves from **A′** to **B′**; and increases, and is positive, while **P** moves from **B′** to **A**. Also,

$$\mathsf{OA} = \mathsf{OA'} = 1.$$

Hence the following results:—

In the first quadrant, as θ increases from 0 to $\frac{1}{2}\pi$,
cos θ decreases from 1 to 0, and is positive.

In the second quadrant, as θ increases from $\frac{1}{2}\pi$ to π,
cos θ decreases from 0 to -1, and is negative.

In the third quadrant, as θ increases from π to $1\frac{1}{2}\pi$,
cos θ increases from -1 to 0, and is negative.

In the fourth quadrant, as θ increases from $1\frac{1}{2}\pi$ to 2π,
cos θ increases from 0 to 1, and is positive.

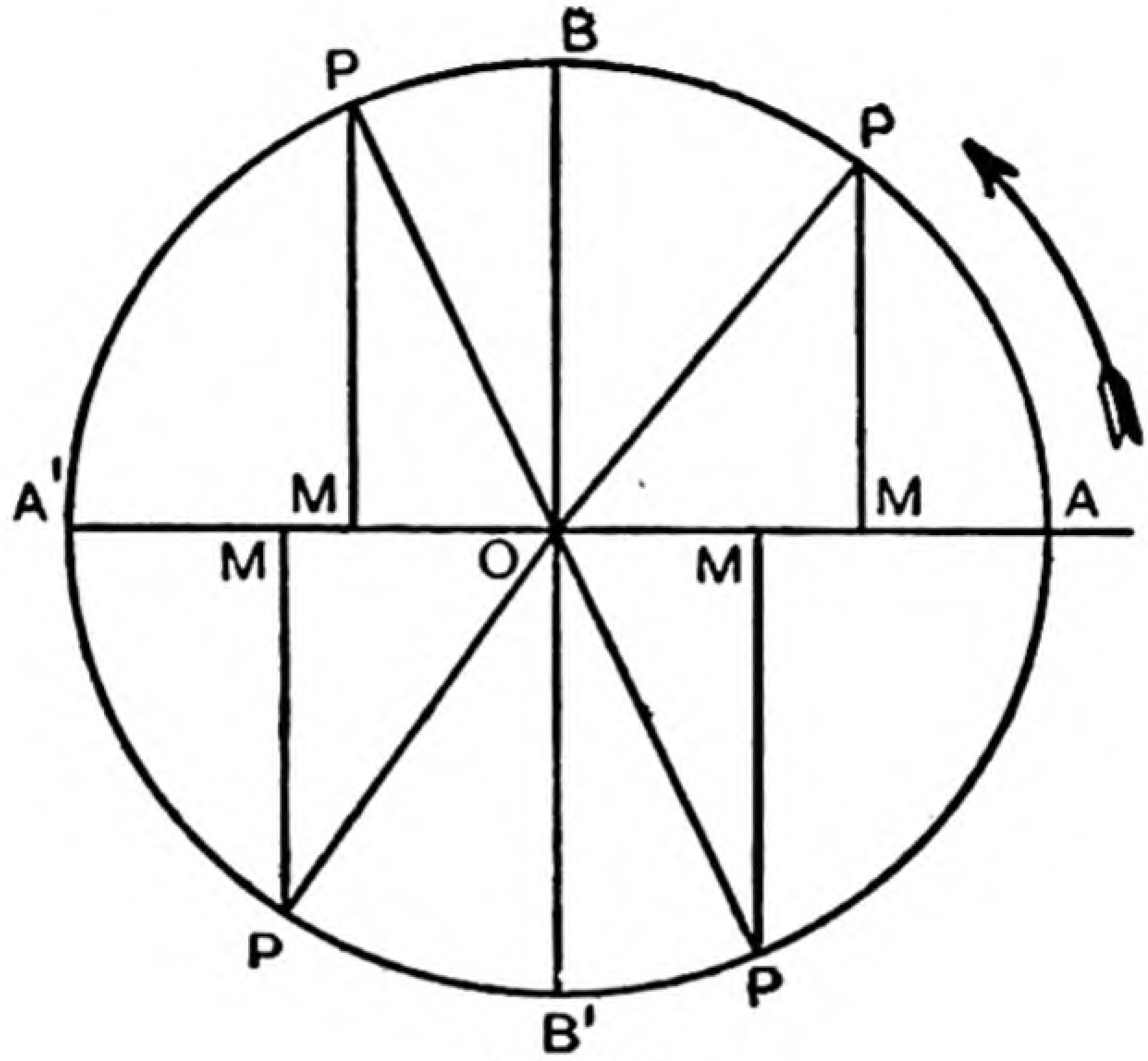


Fig. 43.

As θ continues to increase, the same cycle of changes repeats itself indefinitely every time θ increases by 2π, just as in the case of the sine.

Hence, cos θ, like sin θ, is said to be a periodic function of θ whose period is 2π, and the same statement is equally applicable to the other trigonometric functions of θ.

53. To represent the variations of sin θ graphically.

Take a circle of unit radius (Fig. 44), and, starting from the primitive line **OA**, divide every quadrant of the circumference into any number of equal parts. From the series of points P_1, P_2, . . . thus obtained, draw perpendiculars on **OA**. Then OP_1, OP_2, . . . make with **OA** a series of angles increasing regularly from 0 to 360°, or 2π, and the corresponding perpendiculars M_1P_1, M_2P_2, . . . represent the sines of these angles. Call Fig. 44 the *auxiliary diagram*.

Now draw a second figure (Fig. 45) by taking a horizontal line **OX**, and on it measure off **OD** containing 2π units of length (say $6\frac{2}{7}$ units, taking $\pi = \frac{22}{7}$). **OD** thus represents the circular measure of 360°.

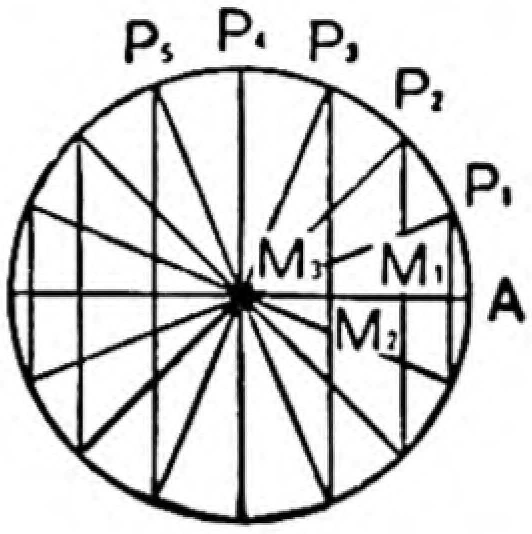


Fig. 44.

Divide **OD** into as many parts as there are divisions in the circumference of the circle in Fig. 44. Then, if $\mathbf{N_1}$, $\mathbf{N_2}$, . . . denote the points of section, the lengths $\mathbf{ON_1}$, $\mathbf{ON_2}$, . . . will represent the circular measures of the angles $\mathbf{AOP_1}$, $\mathbf{AOP_2}$ in Fig. 44.

Through $\mathbf{N_1}$, $\mathbf{N_2}$, . . . erect perpendiculars $\mathbf{N_1Q_1}$, $\mathbf{N_2Q_2}$, . . . equal in length to the corresponding perpendiculars $\mathbf{M_1P_1}$, $\mathbf{M_2P_2}$, . . . in Fig. 44, and drawn in the same directions.

Thus $\mathbf{N_1Q_1}$, $\mathbf{N_2Q_2}$, . . . will represent the sines of the angles whose circular measures are represented by $\mathbf{ON_1}$, $\mathbf{ON_2}$, . . .

By joining up the series of points **O**, $\mathbf{Q_1}$, $\mathbf{Q_2}$, . . . we obtain a curve in which abscissae represent angles, and the corresponding ordinates represent the sines of these angles.

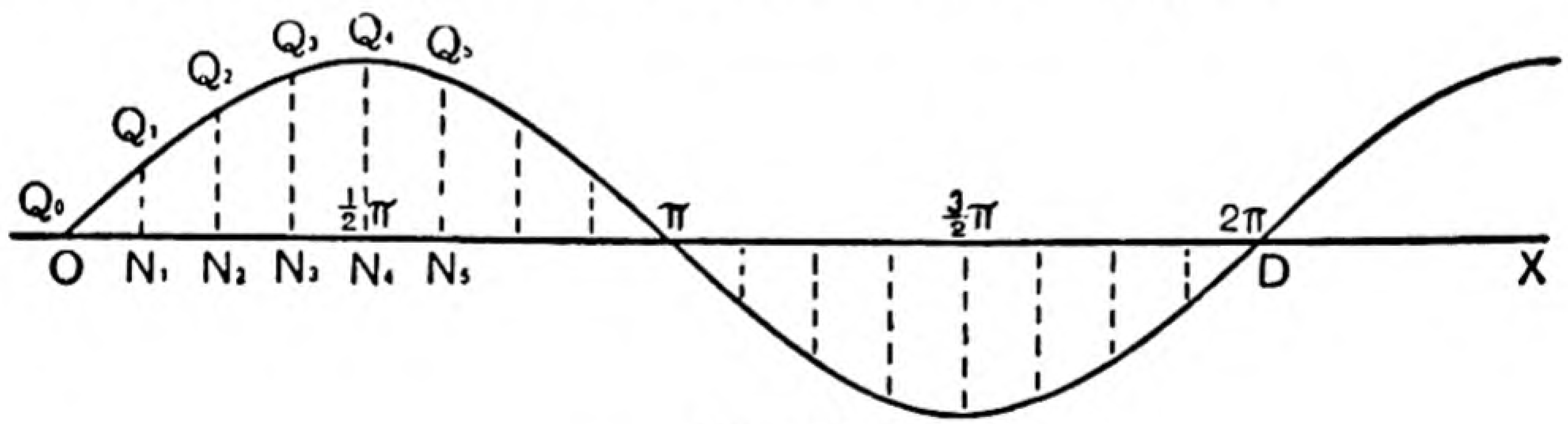


Fig. 45.

This curve is called the **sine curve** or **curve of sines.** By considering the sines of negative angles or of angles greater than four right angles, we see that the complete curve of sines consists of an infinite series of waves extending indefinitely to the left and right of **O**, and all exactly of the same form. Those below **OD** will represent negative sines of angles.

54. To represent the variations of cos θ graphically.

The construction is at first the same as in the last articles but in the second figure the perpendiculars or ordinate,

through the points of section of **OX** (including **O**) must be made equal in length, respectively, to the *horizontal* lengths **OA**, $\mathbf{OM_1}$, $\mathbf{OM_2}$, ... of the auxiliary diagram (Fig. 44), and drawn upwards or downwards according as the latter are to the right or left of **O**. The curve determined by the extremities of these perpendiculars is called the **cosine curve.**

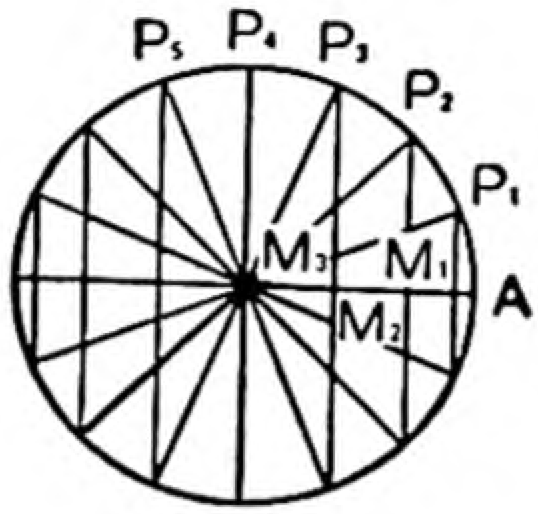


Fig. 44.

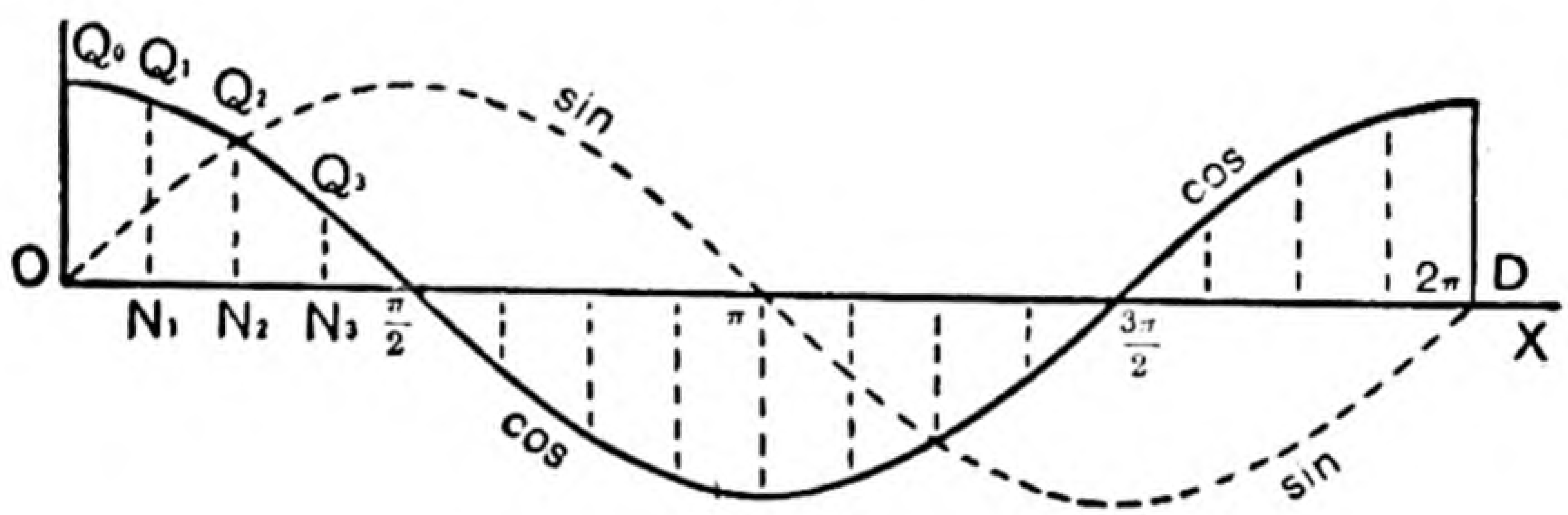


Fig. 46.

The part of the cosine curve corresponding to angles between 0 and 2π is indicated by the continuous line in Fig. 46, the dotted line indicating the sine curve. In the complete cosine curve, the same form repeats itself indefinitely both above and below **OD**.

It will be seen that the cosine curve is simply the sine curve shifted through a distance $\frac{1}{2}\pi$ towards the left.

55. Alternative method of roughly drawing the sine or cosine curves.

Knowing the variations in sign and magnitude of $\sin\theta$ or $\cos\theta$ (§§ 51, 52), it is not difficult, from the considerations mentioned in § 46, to draw a curve representing roughly these fluctuations.

Thus, since sin θ increases from 0 to 1 as θ increases from 0 to $\frac{1}{2}\pi$, the sine curve cuts the horizontal axis at $\theta = 0$, and rises to a distance 1 above it at $\theta = \frac{1}{2}\pi$. Similarly between $0 = \frac{1}{2}\pi$ and $\theta = \pi$ the sine curve descends to the horizontal axis. From $0 = \pi$ to $\theta = 2\pi$, sin θ is negative; hence the sine curve is below the horizontal axis; and so on. Similar considerations apply to other cases.

[The diagrams thus drawn would be sufficiently accurate for examination purposes. Care should be taken to draw all ordinates in their proper directions.]

Abbreviation of the Process.—In any case, when the portion of the curve from $\theta = 0$ to $\theta = \frac{1}{2}\pi$ has been accurately drawn, the *shape* of the remainder may be reproduced from considerations of symmetry.

Ex.—To represent graphically the variations of vers θ.

$$\text{vers } \theta = 1 - \cos\theta.$$

Draw **IH** parallel to the horizontal axis, and at a unit distance above it. Through any point **M** on **IH**, draw the perpendicular **MP** equal to

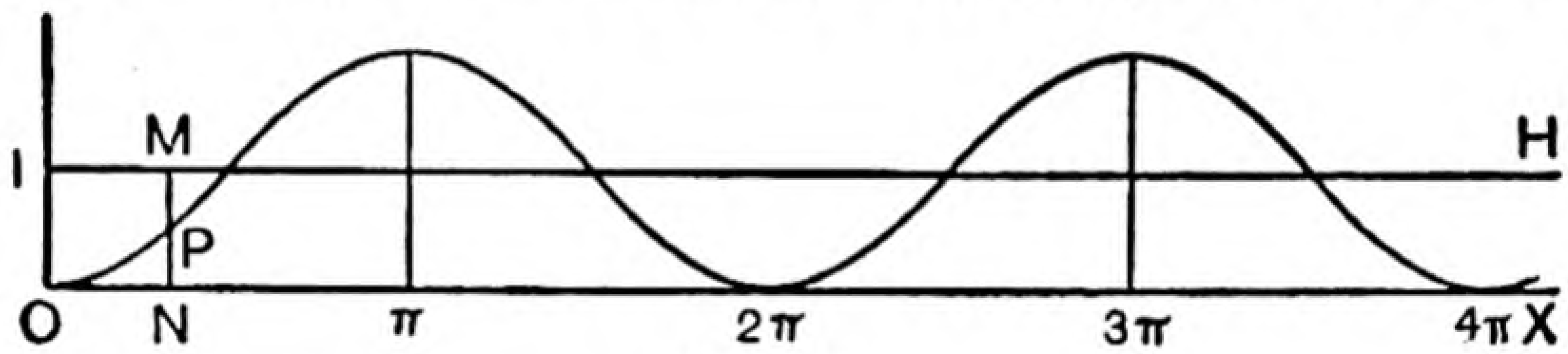


Fig. 47.

the corresponding ordinate in the cosine curve, but in the opposite direction. Produce **MP** to meet **OX** in **N**.

Then, if **ON** or **IM** $= \theta$, we shall have

$$NM = OI = 1, \text{ and } MP = -\cos\theta;$$

hence $NP = 1 - \cos\theta = \text{vers } \theta$, and the locus of **P** is therefore the required versed-sine curve with respect to **OX** as horizontal axis.

The construction shows that this curve is merely the cosine curve reversed in direction (" turned upside down ") and raised through a height unity from the horizontal axis.

Illustrative Exercise.

Draw the cosine curve from $\theta = 0$ to $\theta = \pi$ by dividing each quadrant in the auxiliary diagram (Fig. 44) into six instead of four equal paits.

56. To trace the variations in the tangent of an angle.

Let $\angle AOP = \theta$. On the primitive line cut off $OA = 1$, and draw the indefinite line **ZAZ′** at right angles to **OA**. Produce **OP** to meet **ZZ′** in **T**. Then

$$AT = OA \tan\theta = \tan\theta \quad (\text{since } OA = 1),$$

and is negative when **T** is below **A**, as at **T′**.

Now, as **OP** revolves in the positive direction, **AT** increases, and may be made greater than any finite length by bringing **OP** sufficiently near to parallelism with **ZZ′**. Also, as soon as **OP** has passed the position **OB** (as at **OP′**), **AT** is negative and **T** begins to approach **A** from an infinite distance below **A**. Hence the following results:—

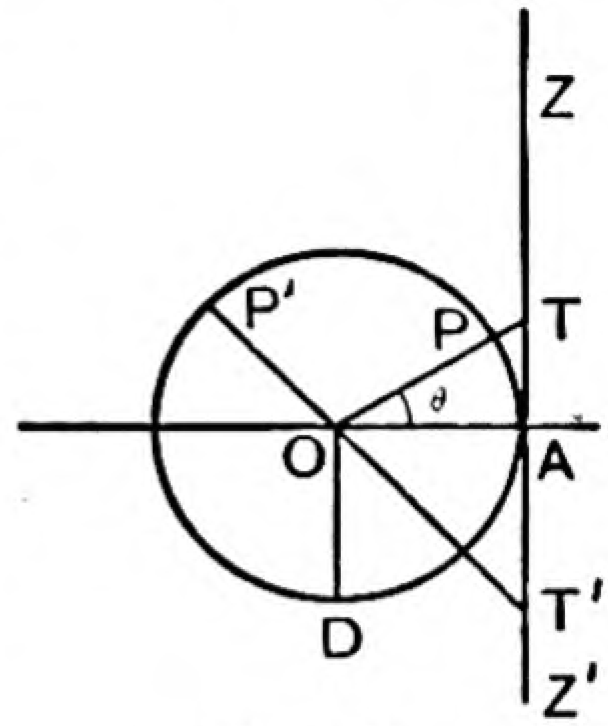


Fig. 48.

In the first quadrant, as θ increases from 0 to $\frac{1}{2}\pi$,

tan θ increases from 0 to $+\infty$, and is positive.

When θ passes through the value $\frac{1}{2}\pi$, tan θ suddenly changes from $+\infty$ to $-\infty$.

In the second quadrant, as θ increases from $\frac{1}{2}\pi$ to π,

tan θ increases (algebraically) from $-\infty$ to 0, and is negative.

In the third quadrant, as θ increases from π to $1\frac{1}{2}\pi$,

tan θ increases from 0 to $+\infty$, and is positive.

When θ passes through the value $1\frac{1}{2}\pi$, tan θ again suddenly changes from $+\infty$ to $-\infty$.

In the fourth quadrant, as θ increases from $1\frac{1}{2}\pi$ to 2π,

tan θ increases from $-\infty$ to 0, and is negative.

The same cycle of changes then repeats itself indefinitely. The period in this case is π, not 2π.

The tangent has the same value when the angle is increased by π.

NOTE.—The tangent is thus continuously increasing except when it becomes infinite, and then it changes suddenly from positive to negative infinity.

The property that a function may change sign in passing through the value infinity should be carefully noted, as it constantly occurs. As an illustration, if a quantity x changes from positive to negative in passing through the value 0, $1/x$ will evidently change from positive to negative, and will become infinite at the changing point when x becomes zero.

57. To trace the variations in the secant of an angle.

In Fig. 48 we have

$$\mathbf{OT} = \mathbf{OA} \sec\theta = \sec\theta \text{ (since } \mathbf{OA} = 1),$$

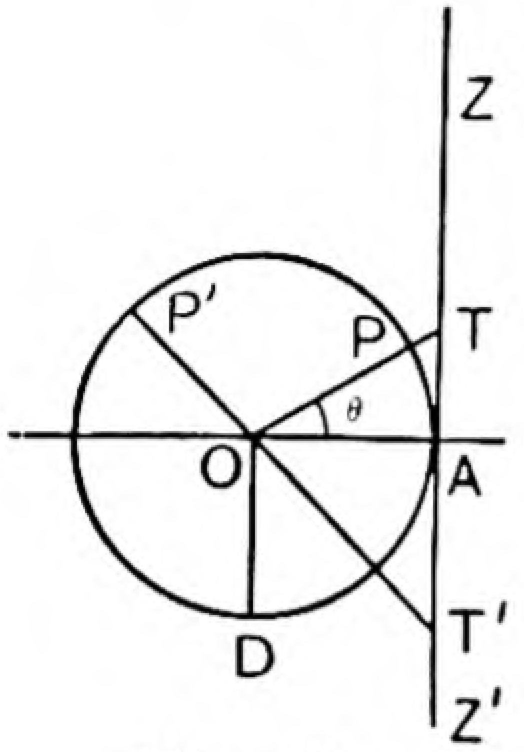


Fig. 48.

sec θ is positive if **OT** lies on the radius **OP**, bounding the angle θ; if **OT** lies on the produced part of this radius on the opposite side of **O**, sec θ is negative; thus, in figure, sec **AOP′** is negative, since it is represented by **OT′** on the opposite side of **O** to **OP′**.

As **OP** revolves round **O**, **OT** at first increases without limit till **OP** becomes parallel to **ZZ′**; **OT** then decreases numerically as **T** moves up to **A** from an infinite distance below, and so on. Hence the following results:—

In the first quadrant, as θ increases from 0 to $\frac{1}{2}\pi$,

sec θ increases from 1 to $+\infty$, and is positive.

When θ passes through the value $\frac{1}{2}\pi$, sec θ suddenly changes from $+\infty$ to $-\infty$.

In the second quadrant, as θ increases from $\frac{1}{2}\pi$ to π,

sec θ increases (algebraically) from $-\infty$ to -1, and is negative.

In the third quadrant, as θ increases from π to $\frac{3}{2}\pi$,

sec θ decreases from -1 to $-\infty$, and is negative.

As θ passes through the value $\frac{3}{2}\pi$, sec θ suddenly changes from $-\infty$ to $+\infty$.

In the fourth quadrant, as θ increases from $\frac{3}{2}\pi$ to 2π,

sec θ decreases from $+\infty$ to $+1$, and is positive,

and so on, the changes repeating themselves in the "period" 2π.

58. To represent the variations of tan θ graphically.

In Fig. 49 (the auxiliary diagram), take a circle of unit radius, and draw radii making with **OA** a series of angles increasing regularly from 0 to 360° or 2π, and produce them to meet the tangent at **A** in $\mathbf{T}_1$, $\mathbf{T}_2$, . . .

Draw **OD** horizontal (Fig. 50), and representing 2π and

divide it into as many parts as there are divisions of the four right angles at **O** in Fig. 49. Then, if through the points of section perpendiculars or ordinates be erected equal in length to AT_1, AT_2, . . ., and drawn in the same direction, these ordinates will represent the tangents of angles whose circular measures are represented by corresponding abscissae, and their extremities will lie along a series of branches representing graphically the variations in the tangent of a variable angle. These branches are indicated by the *thick* lines in Fig. 50. The whole series of branches which repeat themselves indefinitely both to the left and to the right of **O** is called the **tangent curve.**

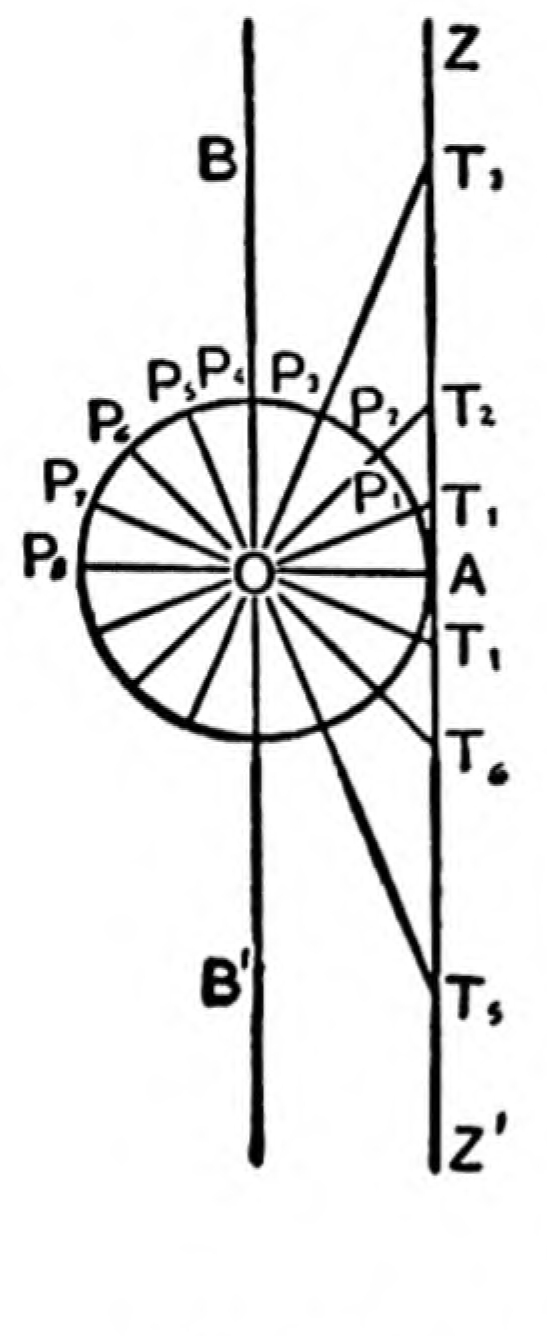


Fig. 49.

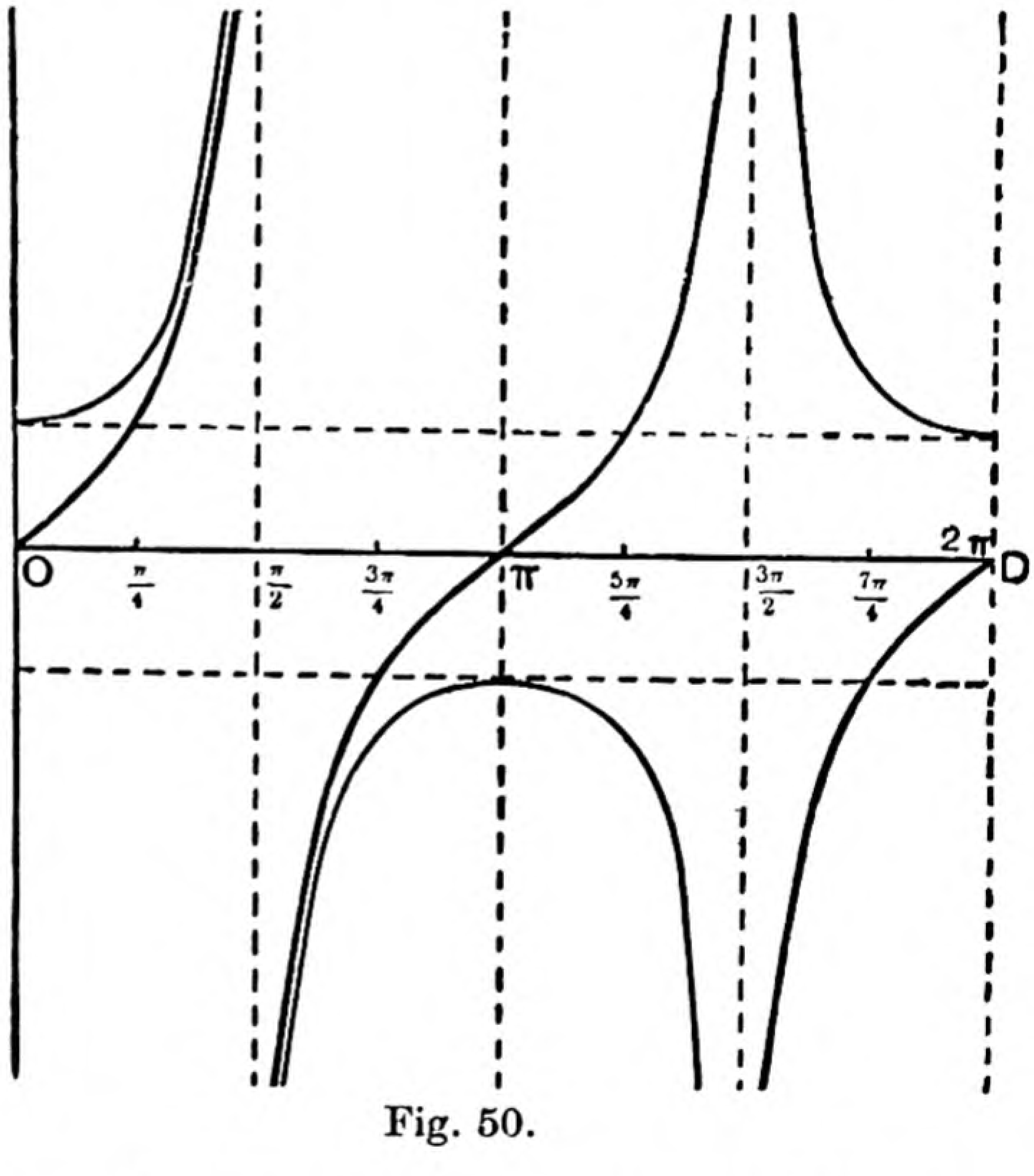


Fig. 50.

The considerations mentioned in §§ 46, 55 give an easy alternative method of roughly reproducing the curve. The same is true of the other curves to be considered in this chapter.

The graph of a function which becomes infinite must, of course, run right off the paper. Thus in Fig. 50, the graph of tan θ must be thought of as continuing upwards to infinity on the left of the ordinate repre-

senting the value $\frac{\pi}{2}$ of θ. As θ passes through the value $\frac{\pi}{2}$, tan θ changes from $+\infty$ to $-\infty$, so that the graph reappears on the right of the ordinate $\theta = \frac{\pi}{2}$ at infinite distance *below* **OD.**

59. To represent the variations of sec θ graphically.

The construction is similar to that for the "tangent curve," but the ordinates in the curve must be measured off equal to the lengths of the secants **OA**, $\mathbf{OT_1}$, $\mathbf{OT_2}$, . . . in Fig. 49.

The ordinates must be taken above or below **OX**, according to the signs of the corresponding secants, and these are determinable from § 57, or from § 39, which tells us that the secant is negative in the second and third quadrants. The series of branches thus obtained may be called the **secant curve**, and is represented by the thin lines in Fig. 50.

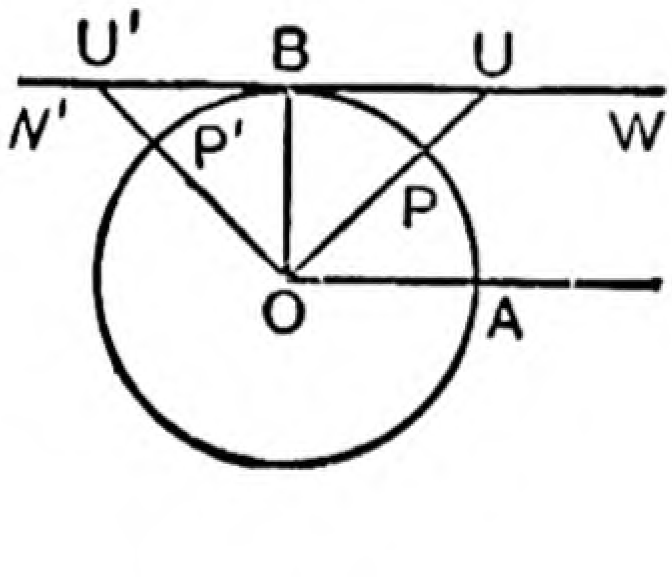


Fig. 51.

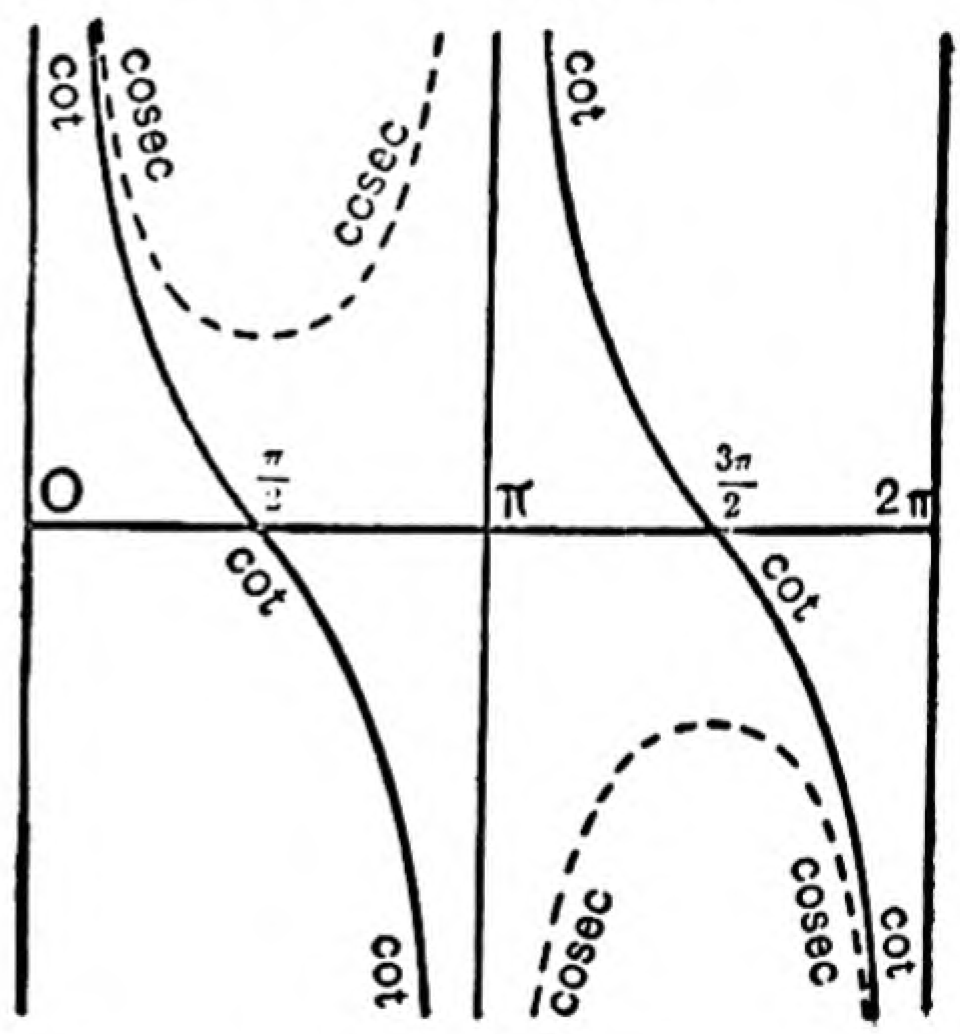


Fig. 52.

60. To trace and represent graphically the variations in the cotangent or cosecant of an angle.

Let $\angle$**AOP** $= \theta$. Draw **OB** at right angles to **OA**, and make it $= 1$. Through **B** draw **WW′** parallel to **OA**, cutting **OP** produced in **U**. Then it will be readily seen that

for the cotangent, **BU** $=$ **OB** cot $\theta =$ cot θ (since **OB** $= 1$);
for the cosecant, **OU** $=$ **OB** cosec $\theta =$ cosec θ „ „ .

The cotangent **BU** is positive when **U** is to the right of **O**, and the cosecant **OU** is positive when **OU** is on the same side of **O** as **P.**

The rest of the work is left to the student as an exercise.

There will now be no difficulty in tracing the variations of cosec θ and cot θ, or in drawing the curves to represent them, taking the ordinates of the **cotangent curve** to be the lengths of **BU**, and those of the **cosecant curve** the lengths of **OU** for a series of angles increasing regularly from 0 to 2π, and giving these ordinates the proper signs. The curves are represented in Fig. 52.

Illustrative Exercises.

(1) Trace fully the changes in the cosecant of a variable angle as that angle increases from 0 to 360°.

(2) Do the same for the cotangent.

Note.—By turning the auxiliary diagram for the tangent and secant (Fig. 49) through a right angle, we see that AT_5, AT_6, . . . represent the cotangents, and OT_5, OT_6, . . . the cosecants of the angles $B'OT_5$, $B'OT_6$, . . . *measured from the line* **OB′**. By bearing this in mind, the ordinates for the cotangent and cosecant curves can be got from Fig. 49, and this was actually done in preparing the figures in the text. The correctness of this construction follows from the relations $\cot\theta = -\tan(\theta - \frac{1}{2}\pi)$, $\operatorname{cosec}\theta = \sec(\theta - \frac{1}{2}\pi)$, which will be proved in Chap. VIII.

EXAMPLES V.

1. Trace the variations in sign and magnitude of the cosine as the angle increases from 0° to 180°.

2. Trace briefly the changes in magnitude and sign of $\sin\theta$ as θ increases from 0° up to 360°.

3. Trace the variations in sign and magnitude of the tangent of an angle as the angle increases from 0° to 180°.

4. Trace the variation in value of cosec θ, as θ changes from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$.

5. Draw the sine curve by the method of § 53, taking 1 in. as the unit of length, for values of θ from 0° to 180°. Find from the graph the angles between 0° and 180° whose sines are respectively $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$. Verify roughly that for each value there are two such angles and their sum is 180°.

6. Draw the cosine curve for values of θ from 0° to 360°. By using tracing paper or otherwise, verify that if the figure be folded about the ordinate corresponding to $\theta = 180°$, the two parts of the curve will coincide. Hence deduce that for any value of θ, $\cos(360° - \theta) = \cos\theta$.

7. Trace the changes in the values of $\cos(\pi \sin\theta)$ as θ varies from 0 to π.

8. Represent graphically the variations of covers θ.

9. By taking the sum of the ordinates of corresponding points on the sine and cosine curves, construct the graph of $(\sin x + \cos x)$ as x varies from 0° to 360°. Find roughly the values of x for which $\sin x + \cos x = 0$.

CHAPTER VI.

TRIGONOMETRIC FUNCTIONS OF CERTAIN ANGLES.

In this chapter we shall determine the values of the trigonometric functions of certain useful angles which constantly occur in problems, and with which the student will require to be familiar.

61. To find the trigonometric functions of an angle of 45°, or $\pi/4$.

Draw a square **ABCD**, and draw the diagonal **AC**.

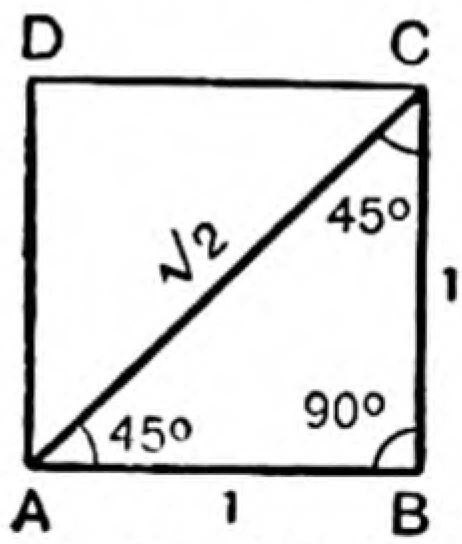


Fig. 53.

$$\begin{aligned} \text{Then} \qquad \angle \mathbf{BAC} &= \text{half a right angle} = 45°, \\ \angle \mathbf{CBA} &= \text{a right angle} \quad = 90°; \\ \therefore \quad \mathbf{AC}^2 &= \mathbf{AB}^2 + \mathbf{BC}^2. \qquad \text{(Euc. I. 47)} \\ \text{Also} \qquad \mathbf{AB} &= \mathbf{BC}; \\ \therefore \quad \mathbf{AC}^2 &= 2\mathbf{AB}^2 \quad = 2\mathbf{BC}^2; \\ \therefore \quad \mathbf{AC} &= \sqrt{2}.\mathbf{AB} = \sqrt{2}.\mathbf{BC}; \end{aligned}$$

$$\left.\begin{aligned} \therefore \quad \sin 45° &= \frac{\mathbf{BC}}{\mathbf{AC}} = \frac{1}{\sqrt{2}}; & \operatorname{cosec} 45° &= \frac{\mathbf{AC}}{\mathbf{BC}} = \sqrt{2} \\ \cos 45° &= \frac{\mathbf{AB}}{\mathbf{AC}} = \frac{1}{\sqrt{2}}; & \sec 45° &= \frac{\mathbf{AC}}{\mathbf{AB}} = \sqrt{2} \\ \tan 45° &= \frac{\mathbf{BC}}{\mathbf{AB}} = 1; & \cot 45° &= \frac{\mathbf{AB}}{\mathbf{BC}} = 1 \end{aligned}\right\} \ldots \textbf{(22)}$$

Of these, only the sine, cosine, and tangent need be remembered, the other three being their reciprocals. The same is the case in the following articles.

COR.—If the angles of a triangle be 45°, 45°, and 90°, the sides are proportional to 1, 1, and $\sqrt{2}$.

[From this corollary, the trigonometric functions of 45° can be readily written down with the help of Fig. 53.]

62. To find the trigonometric functions of an angle of 30°, or $\pi/6$.

Draw an equilateral triangle **ABC**. Join **A** to **D**, the middle point of **BC**. Then the triangles **ABD, ACD** are equal in every respect.

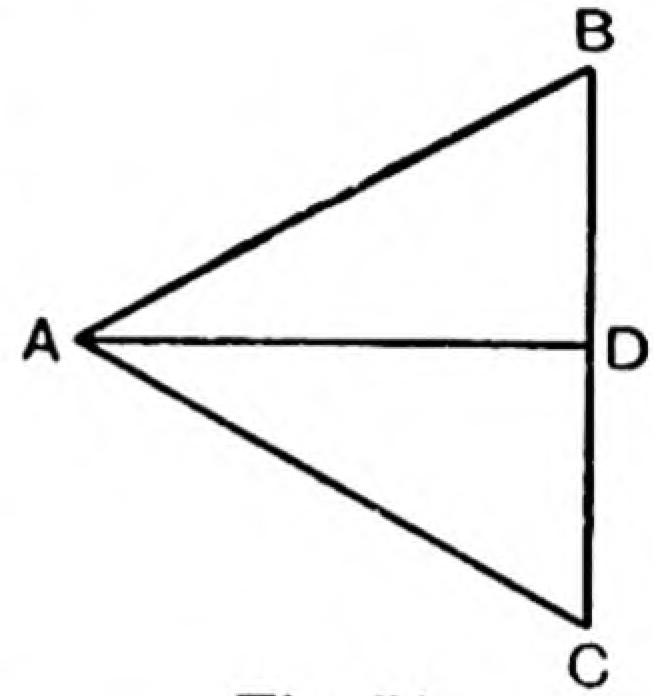


Fig. 54.

But the three angles of an equilateral triangle are all equal, and together = two right angles = 180°; therefore each = 60°;

$$\therefore \quad \angle \mathbf{DBA} = 60^\circ;$$

$$\therefore \quad \angle \mathbf{DAB} = 30^\circ.$$

Also
$$\angle \mathbf{ADB} = \angle \mathbf{ADC} = 90^\circ;$$

$$\therefore \quad \mathbf{AD}^2 + \mathbf{DB}^2 = \mathbf{AB}^2. \qquad \text{(Euc. I. 47)}$$

But
$$\mathbf{AB} = \mathbf{CB} = 2\mathbf{DB};$$

$$\therefore \quad \mathbf{AD}^2 + \mathbf{DB}^2 = 4\mathbf{DB}^2, \quad \text{or} \quad \mathbf{AD}^2 = 3\mathbf{DB}^2;$$

$$\therefore \quad \mathbf{AD} = \sqrt{3}\,.\,\mathbf{DB};$$

$$\therefore \left.\begin{aligned} \sin 30^\circ &= \frac{\mathbf{DB}}{\mathbf{AB}} = \frac{1}{2}; \qquad & \operatorname{cosec} 30^\circ &= \frac{\mathbf{AB}}{\mathbf{DB}} = 2 \\ \cos 30^\circ &= \frac{\mathbf{AD}}{\mathbf{AB}} = \frac{\sqrt{3}}{2}; & \sec 30^\circ &= \frac{\mathbf{AB}}{\mathbf{AD}} = \frac{2}{\sqrt{3}} \\ \tan 30^\circ &= \frac{\mathbf{DB}}{\mathbf{AD}} = \frac{1}{\sqrt{3}}; & \cot 30^\circ &= \frac{\mathbf{AD}}{\mathbf{DB}} = \sqrt{3} \end{aligned}\right\} \ldots \textbf{(23)}$$

63. To find the trigonometric functions of an angle of 60°, or $\pi/3$.

Take the figure of § 62 and turn it round (Fig. 55).

Then
$$\angle \mathbf{CBA} = 60^\circ;$$

$$\left.\begin{aligned} \therefore\quad \sin 60^\circ &= \frac{\mathbf{DA}}{\mathbf{AB}} = \frac{\sqrt{3}}{2}; & \operatorname{cosec} 60^\circ &= \frac{\mathbf{AB}}{\mathbf{DA}} = \frac{2}{\sqrt{3}} \\ \cos 60^\circ &= \frac{\mathbf{BD}}{\mathbf{AB}} = \frac{1}{2}; & \sec 60^\circ &= \frac{\mathbf{AB}}{\mathbf{BD}} = 2 \\ \tan 60^\circ &= \frac{\mathbf{DA}}{\mathbf{BD}} = \sqrt{3}; & \cot 60^\circ &= \frac{\mathbf{BD}}{\mathbf{DA}} = \frac{1}{\sqrt{3}} \end{aligned}\right\} \quad \ldots \text{(24)}$$

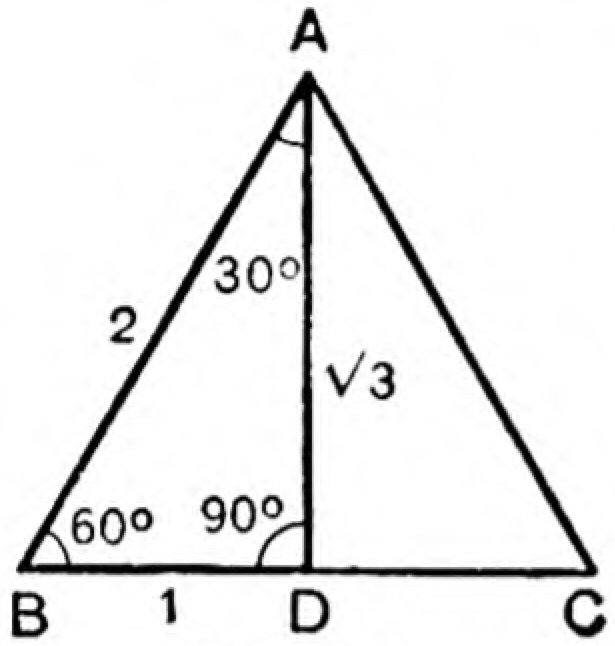


Fig. 55.

COR.—If the angles of a triangle be 30°, 60°, and 90°, the sides opposite these angles are proportional to 1, $\sqrt{3}$, and 2.

From this corollary the trigonometric functions of both 30° and 60° can be readily written down with the help of Fig. 55.

64. To find the trigonometric functions of an angle of 0°.

Let **BOC** be a very small angle. Then, if **CB** be drawn perpendicular on **OB**, the abscissa **OB** and radius **OC** will be very nearly equal, while the ordinate **BC** will be small compared with either of them.

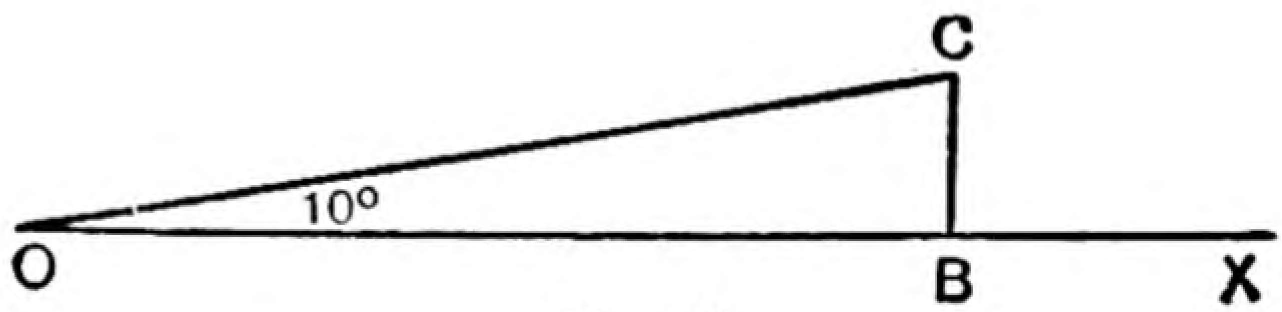


Fig. 56.

If now the angle **BOC** becomes *zero*, **OC** will coincide with **OB**, and **C** with **B**, and the perpendicular **BC** will vanish ;

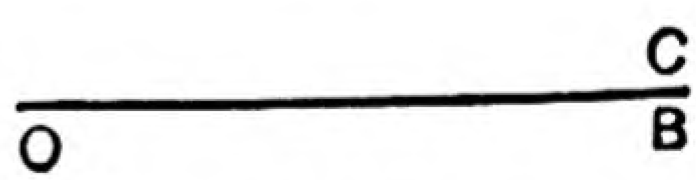


Fig. 57.

$$\therefore\quad \mathbf{OB} = \mathbf{OC} \quad \text{and } \mathbf{BC} = 0;$$

$$\left.\begin{aligned} \therefore\quad \sin 0^\circ &= \frac{\mathbf{BC}}{\mathbf{OC}} = 0; & \operatorname{cosec} 0^\circ &= \frac{\mathbf{OC}}{0} = \infty \\ \cos 0^\circ &= \frac{\mathbf{OB}}{\mathbf{OC}} = 1; & \sec 0^\circ &= \frac{\mathbf{OC}}{\mathbf{OB}} = 1 \\ \tan 0^\circ &= \frac{\mathbf{BC}}{\mathbf{OB}} = 0; & \cot 0^\circ &= \frac{\mathbf{OB}}{0} = \infty \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(25)}$$

65. To find the trigonometric functions of an angle of 90°, or $\pi/2$.

Let **DAC** be very nearly equal to 90°. Then, if **CB** be drawn perpendicular on **AD**, the radius **AC** and ordinate **BC** will be nearly equal, while the abscissa **AB** will be small compared with either. And, if **DAC** becomes exactly 90°, **CB** will coincide with **CA**, and **B** with **A**.

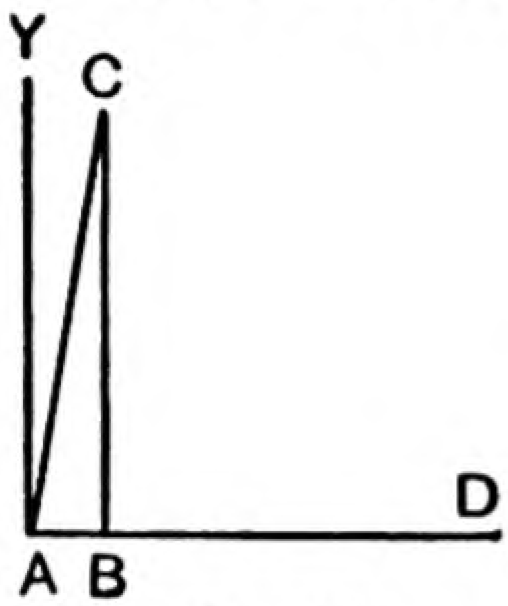


Fig. 58.

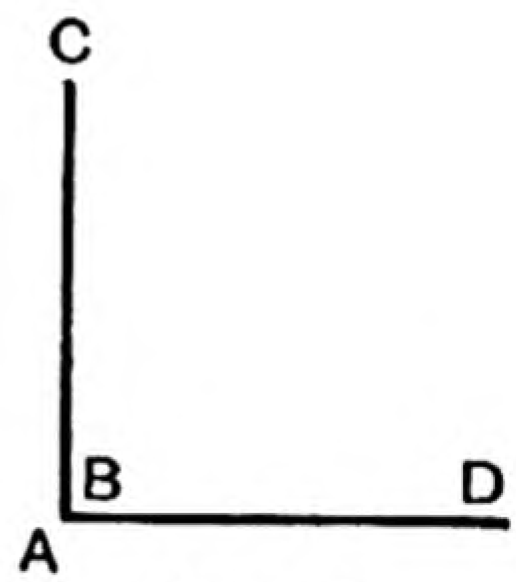


Fig. 59.

$$\therefore \quad \mathbf{BC} = \mathbf{AC} \quad \text{and} \quad \mathbf{AB} = 0;$$

$$\left.\begin{aligned}
\therefore \quad \sin 90^\circ &= \frac{\mathbf{BC}}{\mathbf{AC}} = 1, & \operatorname{cosec} 90^\circ &= \frac{\mathbf{AC}}{\mathbf{BC}} = 1 \\
\cos 90^\circ &= \frac{\mathbf{AB}}{\mathbf{AC}} = 0, & \sec 90^\circ &= \frac{\mathbf{AC}}{\mathbf{AB}} = \frac{\mathbf{AC}}{0} = \infty \\
\tan 90^\circ &= \frac{\mathbf{BC}}{\mathbf{AB}} = \frac{\mathbf{BC}}{0} = \infty, & \cot 90^\circ &= \frac{\mathbf{AB}}{\mathbf{BC}} = \frac{0}{\mathbf{BC}} = 0
\end{aligned}\right\}$$

.... (26)

66. The values of the three principal trigonometric functions of the above angles should be remembered.

To save trouble in fixing these in the memory, it may be noticed that, for the common angles

$$0^\circ, \qquad 30^\circ, \qquad 45^\circ, \qquad 60^\circ, \qquad 90^\circ,$$

the sines are the square roots of

$$\frac{0}{4}, \qquad \frac{1}{4}, \qquad \frac{2}{4}, \qquad \frac{3}{4}, \qquad \frac{4}{4},$$

the cosines are the square roots of

$$\frac{4}{4}, \qquad \frac{3}{4}, \qquad \frac{2}{4}, \qquad \frac{1}{4}, \qquad \frac{0}{4};$$

and each *tangent* is the corresponding *sine* ÷ *cosine*.

Observe that the series of values of the cosines is the same as the series for sines written backwards.

NOTE.—It is usual to write $\tan 90° = \infty$ as in (26). We have $\tan \mathbf{CAB} = \frac{\mathbf{BC}}{\mathbf{AB}}$, and thus as in § 48 when **AB** approaches 0, tan **CAB** approaches ∞. We are, however, taking **CAB** as less than 90° in Fig. 58; **B** lies to the right of **A** and **AB** is positive. If we took the angle **DAC** a little greater than a right angle, **B** would fall on the other side of **A** and the tangent would be negative, and numerically very large. In this case, tan **CAD** approaches $-\infty$ when $\angle$ **CAD** becomes a right angle. Actually, as we saw in § 56, when θ passes through the value 90° or $\frac{\pi}{2}$, $\tan \theta$ may be $+\infty$ or $-\infty$ according as we consider angles a little less than 90° or a little greater than 90°.

Similarly in other cases in the formulae 25-29, where ∞ is given as the value of one of the functions, there is actually an abrupt change from $+\infty$ to $-\infty$, as shown in the corresponding graphs of Chap. V.

67. To find the trigonometric functions of an angle of 180°, or π radians.

Let $\angle$ **XOP** be nearly equal to 180°. Then, if **PM** be drawn perpendicular to **XO** produced, the ordinate **MP** will be small

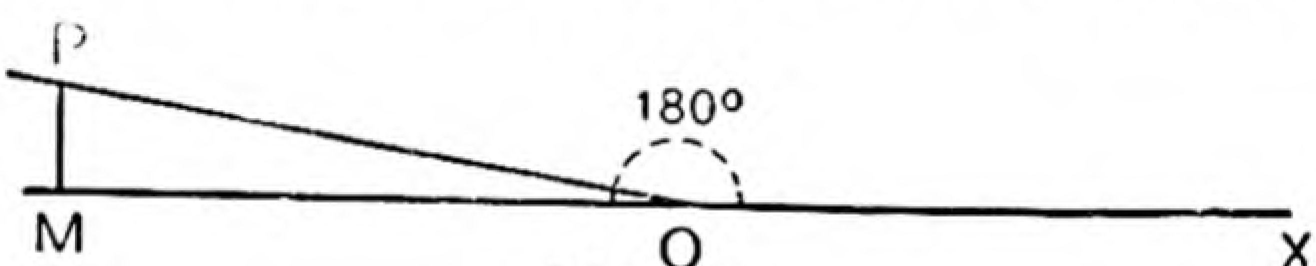


Fig. 60.

and the abscissa **OM** will be nearly equal in length to the radius **OP**; but the former* will be negative, while the radius is always positive. Hence, when $\angle$ **XOP** actually becomes equal to 180°, we have

$$\mathbf{MP} = 0, \quad \mathbf{OM} = -\mathbf{OP};$$

$$\left.\begin{aligned} \therefore \quad \sin 180° &= \frac{\mathbf{MP}}{\mathbf{OP}} = 0; & \operatorname{cosec} 180° &= \frac{\mathbf{OP}}{0} = \infty \\ \cos 180° &= \frac{\mathbf{OM}}{\mathbf{OP}} = -1; & \sec 180° &= \frac{\mathbf{OP}}{\mathbf{OM}} = -1 \\ \tan 180° &= \frac{\mathbf{MP}}{\mathbf{OM}} = 0; & \cot 180° &= \frac{\mathbf{OM}}{0} = \infty \end{aligned}\right\} \ldots (27)$$

* *I.e.* **OM**.

68. To find the trigonometric functions of an angle of 270°, or $3\pi/2$ radians.

Let the radius vector revolve from **OX** to **OP** through an angle very nearly equal to 270°. Then the abscissa **OM** will be small, and the ordinate **MP** very nearly equal to the radius **OP**, but opposite in sign because **P** is below **M**. Hence, when the angle actually becomes equal to 270°, we have

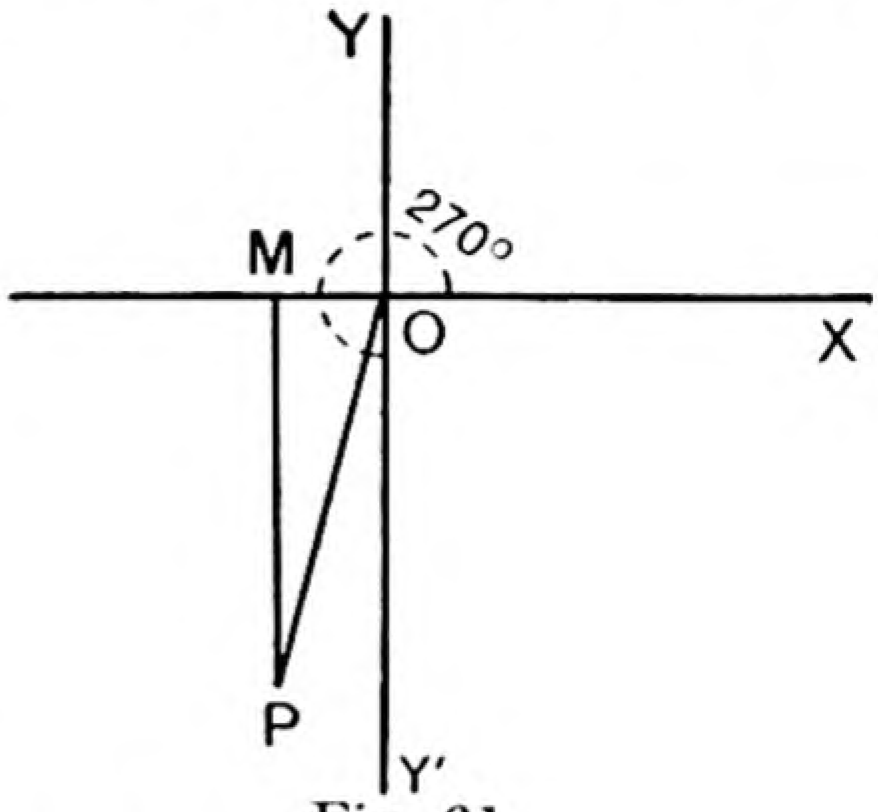


Fig. 61.

$$\mathbf{MP} = -\mathbf{OP}, \quad \mathbf{OM} = 0;$$

$$\left.\begin{aligned} \therefore\quad \sin 270^\circ &= \frac{\mathbf{MP}}{\mathbf{OP}} = -1; & \operatorname{cosec} 270^\circ &= \frac{\mathbf{OP}}{\mathbf{MP}} = -1 \\ \cos 270^\circ &= \frac{\mathbf{OM}}{\mathbf{OP}} = 0; & \sec 270^\circ &= \frac{\mathbf{OP}}{\mathbf{OM}} = \infty \\ \tan 270^\circ &= \frac{\mathbf{MP}}{\mathbf{OM}} = \infty; & \cot 270^\circ &= \frac{\mathbf{OM}}{\mathbf{MP}} = 0 \end{aligned}\right\} \ldots (28)$$

69. To find the trigonometric functions of 360°, or 2π radians.

When the radius vector has revolved through 360°, it will have returned to its original position and will be where it would have been if it had not revolved at all.

Hence the trigonometric functions of 360° *are the same as of* 0°, *viz.*—

$$\left.\begin{aligned} \sin 360^\circ &= 0, & \operatorname{cosec} 360^\circ &= \infty \\ \cos 360^\circ &= 1, & \sec 360^\circ &= 1 \\ \tan 360^\circ &= 0, & \cot 360^\circ &= \infty \end{aligned}\right\} \ldots\ldots\ldots (29)$$

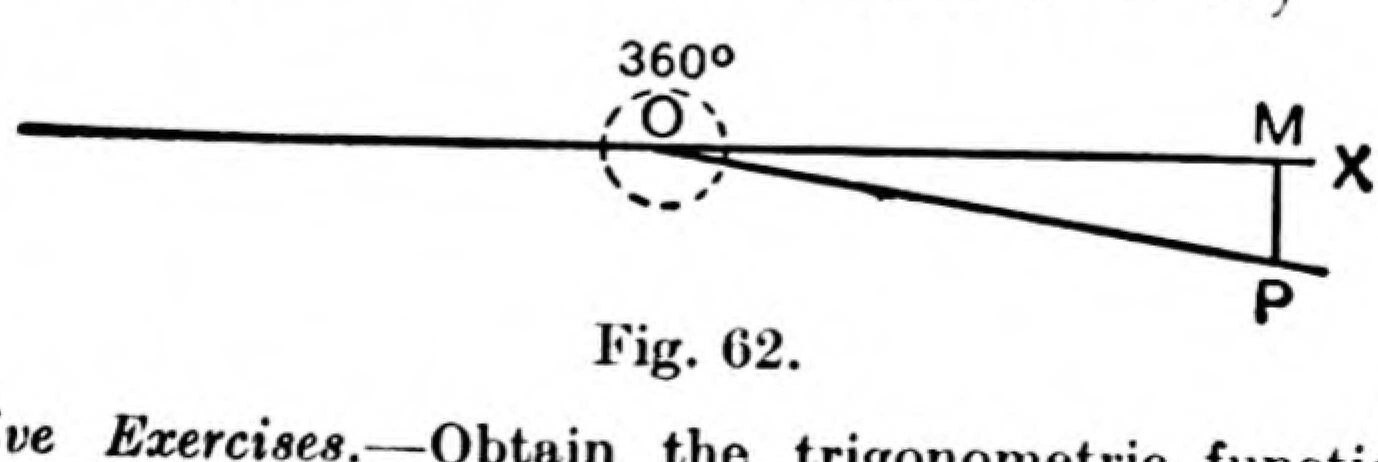


Fig. 62.

Illustrative Exercises.—Obtain the trigonometric functions of 360° from a figure without assuming that they are the same as those of 0°.

70. Table.—The results of the last few articles (§§ 64-69) enable us to exhibit in the following tabular form the values of the trigonometric functions for angles bounding the several quadrants and the signs of the functions in these quadrants:—

Angle ...	0°		90°		180°		270°		360°
sine ...	**0**	+	**1**	+	**0**	−	**−1**	−	**0**
cos	**1**	+	**0**	−	**−1**	−	**0**	+	**1**
tan	**0**	+	**∞**	−	**0**	+	**∞**	−	**0**
cot	**∞**	+	**0**	−	**∞**	+	**0**	−	**∞**
sec	**1**	+	**∞**	−	**−1**	−	**∞**	+	**1**
cosec ...	**∞**	+	**1**	+	**∞**	−	**−1**	−	**∞**
∠ in radians	0		$\frac{1}{2}\pi$		π		$\frac{3}{2}\pi$		2π

71. Applications to heights and distances.—We shall now give a few further applications of trigonometrical notation, many of them assuming a knowledge of the trigonometric functions whose values have just been found.

Ex. The altitude of a tower is 30° at the end of a horizontal base of 100 yd. from its foot. Find the height of the tower in feet.

$$\frac{h}{300} = \tan 30° = \frac{1}{\sqrt{3}},$$

$$\therefore \quad h = \frac{300}{\sqrt{3}} = 300 \times \frac{\sqrt{3}}{3} = 100\sqrt{3} \text{ ft.}$$

N.B.—*Always rationalise the denominators of surds.*

NOTE.—Although problems in heights and distances are often proposed for solution in which the observed angles are 30°, 45°, or 60°, *it must not be inferred that the angles would be likely to have these values in any actual observation.*

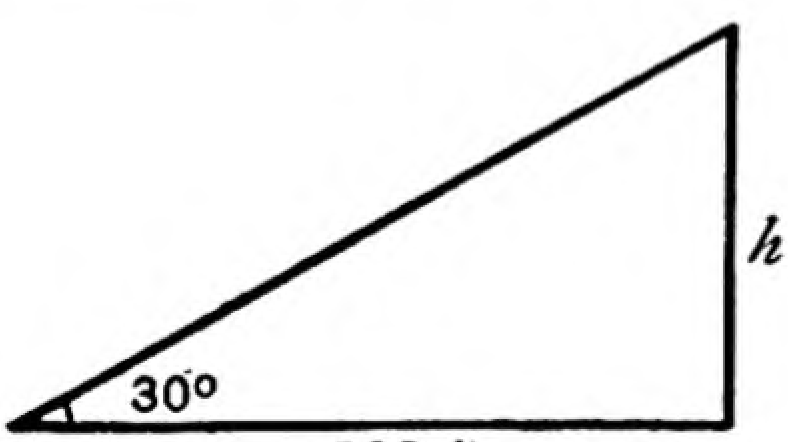


Fig. 63.

Thus it would be extremely improbable for an observer to select the place of observation so as to make the elevation of a church tower exactly = 30°. Such problems, however, afford valuable exercises in the use of trigonometric functions.

72. To find the height of an object standing on a horizontal plane, when the base of the object is inaccessible.

Suppose **QP** to be the object, **P** being inaccessible, and

let **A, B** be two accessible points in a horizontal line through **P**.

The length **AB**, and the angles **QAB, QBP** are observed. Call them a, A, B.

Let $\mathbf{PQ} = x$.

Then $\mathbf{AP} = x \cot A$,

$\mathbf{BP} = x \cot B$.

Subtracting,

$$a = x(\cot A - \cot B),$$

$$x = \frac{a}{\cot A - \cot B}.$$

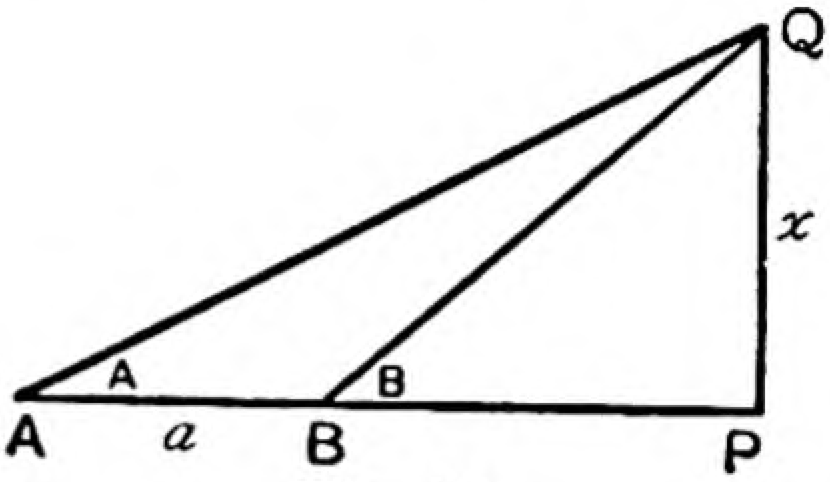


Fig. 64.

NOTE.—This result should not be remembered, but the method used. The method is very often used to determine the height of a mountain.

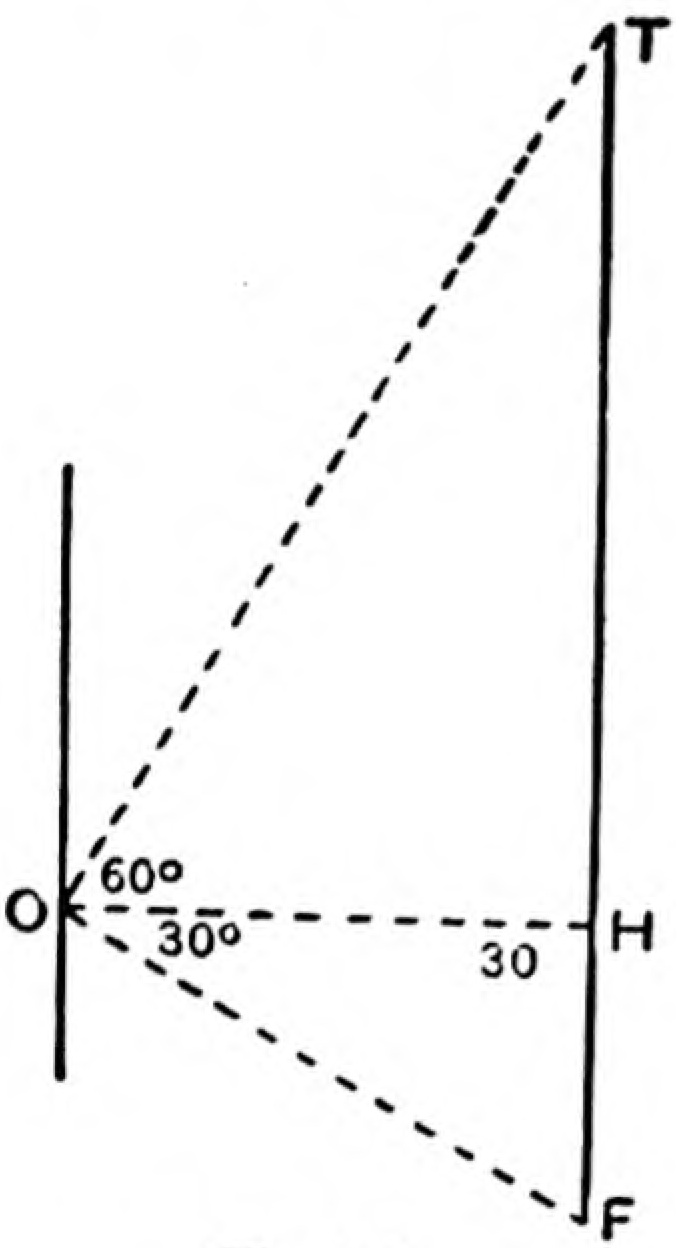


Fig. 65.

The following example is also instructive:—

Ex. The height of a house subtends a right angle at an opposite window, the top being 60° above the horizontal straight line. Find the height of the house, the street being 30 ft. wide.

By figure,

$$\mathbf{HT} = 30 \tan 60° = 30\sqrt{3},$$

$$\mathbf{HF} = 30 \tan 30° = 30\,\frac{1}{\sqrt{3}}$$

$$= 30\,\frac{\sqrt{3}}{3} = 10\sqrt{3};$$

$$\therefore\quad \mathbf{FT} = \mathbf{HT} + \mathbf{FH}$$

$$= 40\sqrt{3} \text{ feet}$$

$$= \text{height of house.}$$

EXAMPLES VI.

1. Prove that $\sin 60° \cos 30° - \cos 60° \sin 30° = \sin 30°$.
2. Prove that $\cos 60° \cos 30° - \sin 60° \sin 30° = 0$.

3. Prove that
$\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$.

4. Prove that $(\sin 30^\circ)^2 + (\cos 30^\circ)^2 = 1$.

5. Prove that $(\tan 30^\circ)^2 = (\sec 30^\circ)^2 - 1$.

6. Prove that $(\sin 60^\circ - \sin 45^\circ)(\cos 30^\circ + \cos 45^\circ) = \frac{1}{4}$.

If $A = 30^\circ$, $B = 45^\circ$, $C = 60^\circ$, find the values of the following expressions (7–16):—

7. $\sin^2 A + \sin^2 C$.

8. $\sin A + \cos^2 B$.

9. $\tan B + \cot B$.

10. $\cos B \sin B - \sin^2 A$.

11. $\dfrac{\sec A}{\tan A} - \dfrac{\sec B}{\cot A}$.

12. $\dfrac{\tan A \tan B + \tan B \tan C + \tan C \tan A}{\tan A + \tan B + \tan C}$.

13. $\dfrac{\sin A \cos B + \cos A \sin B}{\sin B \cos C - \cos B \sin C}$.

14. $\dfrac{\cos A \cos B + \sin A \sin B}{\cos B \cos C + \sin B \sin C}$.

15. $\dfrac{2 \tan A}{1 - \tan^2 A} - \tan C$.

16. $3 \sin A - 4 \sin^3 A$.

If $A = 90^\circ$, $B = 60^\circ$, $C = 45^\circ$, $D = 30^\circ$, $E = 0^\circ$, find the values of the following expressions (17–23), and verify the relations (24–29):—

17. $\sin^2 A + \sin^2 B + \sin^2 C + \sin^2 D + \sin^2 E$.

18. $\tan A \tan B \tan C$.

19. $\cot A \cot C + \cos B \cos E$.

20. $\dfrac{\sin A}{\cos B} + \dfrac{\tan C}{\cot D} + \dfrac{\sec E}{\operatorname{cosec} B}$.

21. $\tan B \tan D - \tan C \tan E$.

22. $\tan^2 B + \tan^2 C + \tan^2 D$.

23. $\dfrac{\cos E \operatorname{cosec} C}{\tan B \sec D}$.

24. $\sin B \cos D + \cos B \sin D = \sin A$.

25. $2 \sin C \cos C = \sin A$.

26. $\sin B \sin E + \cos B \cos E = \cos B$.

27. $4(\cos D)^3 - 3 \cos D = \cos 3D$.

28. $\sin 3D = 3 \sin D - 4(\sin D)^3$

29. $2 \sin \frac{1}{2}B = \sqrt{1 + \sin B} - \sqrt{1 - \sin B}$.

If $\alpha = 0^\circ$, $\beta = \pi/6$, $\gamma = \pi/4$, $\delta = \pi/3$, $\theta = \pi/2$, find the values of the following expressions (30–34):—

30. $\cos \alpha \sin \gamma \cos \delta$.

31. $\sin \theta \cos \dfrac{\pi}{4} \operatorname{cosec} \delta$.

32. $(\sin \delta - \sin \gamma)(\cos \beta + \cos \gamma)$.

33. $\tan^2 \delta - \tan^2 \beta$.

34. $\dfrac{\sin^2 \delta - \sin^2 \beta}{\cos^2 \delta - \cos^2 \beta}$.

35. Prove that $\sin^2 30^\circ : \sin^2 45^\circ : \sin^2 60^\circ : \sin^2 90^\circ$ as $1 : 2 : 3 : 4$.

36. **ABC** is a triangle right-angled at **A** and having the angle $B = 30^\circ$. **AD** is drawn perpendicular to **BC** and is 10 ft. in length. Find the length of the sides of the triangle.

37. Find the length of the shadow of a stick 6 ft. high, when the sun is at an altitude of 30°.

38. At what angle must the stick in Question 37 be inclined to the ground in order that the length of its shadow may be the greatest possible? Show that length of shadow is then twice length of stick.

39. A church tower is surmounted by a spire. At the distance of 30 ft. from the tower the elevation of the top of the spire is 45°, and of the tower 30°. What is the height of the spire?

40. The elevation of a tower from a point due south of it is 45°; if the observer move a hundred yards to the east, the elevation is 30°. Find the height of the tower.

41. A wall 12 ft. high runs east and west. What is the distance from the wall at which a man 6 ft. high can just see the sun at noon when its elevation is 30°?

42. A man on a ship at sea going 30° W. of N. sees a lighthouse due north. After sailing 5 miles he sees it due east. How far was he from the lighthouse on each occasion?

43. A balloon due west at noon was travelling horizontally due south at the rate of 10 miles an hour. Half an hour later its elevation had fallen from 60° to 30°. How high was it above the ground?

44. A ladder 20 ft. long reaches to a distance 20 ft. from the top of a flagstaff. At the foot of the ladder the elevation of the top of the staff is 60°. Find the height of the flagstaff.

45. The angle of elevation of the top of a cliff is observed to be 60°; 100 yd. farther out to sea it is observed to be 30°. Find height of cliff.

46. A target is 6 ft. high and 8 ft. broad. Find the tangents of the angles which its four edges subtend at a point 100 ft. in front of its left-hand lower corner.

47. A man stands at a point **A** on the bank **AB** of a straight river, and observes that the line joining **A** to a post **C** on the opposite bank makes with **AB** an angle of 30°. He then goes 200 yd. along the bank to **B**, and finds that **BC** makes an angle of 60° with the bank. Find the breadth of the river.

48. A mountain is observed to be due south and to have an elevation of 60°. On going a mile to the east its altitude is seen to be only 30°. Find its height.

49. From the top of a hill the angles of depression of the top and bottom of a flagstaff 30 ft. high are observed to be 30° and 31°. Given $\tan 31° = \cdot 6009$, find the height of the hill.

50. A man on the top of a cliff 300 ft. high observes two boats due east, one beyond the other. Their angles of depression are 30° and 40°. Find their distance apart.

51. Determine the height of a chimney when it is found that walking towards it 100 ft. in a horizontal line through the base changes the angular elevation of the top from 30° to 60°.

CHAPTER VII.

RELATIONS BETWEEN THE TRIGONOMETRIC FUNCTIONS OF THE SAME ANGLE.

73. In Chapter IV. we proved the relations (16)—(18)—

$$\operatorname{cosec} A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A};$$

$$\cot A = \frac{1}{\tan A}.$$

We shall now establish certain other formulae connecting the various trigonometric functions of an angle, and shall prove that, if the value of *one* of these functions be known, the others can all be found.

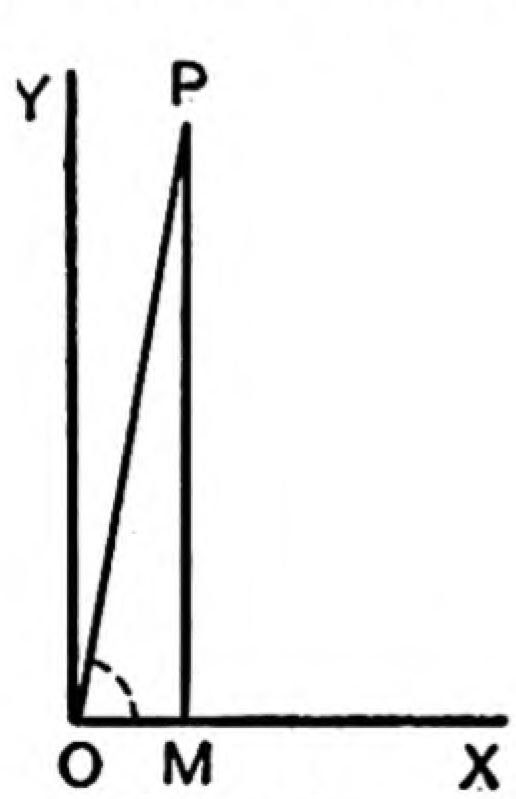


Fig. 66.

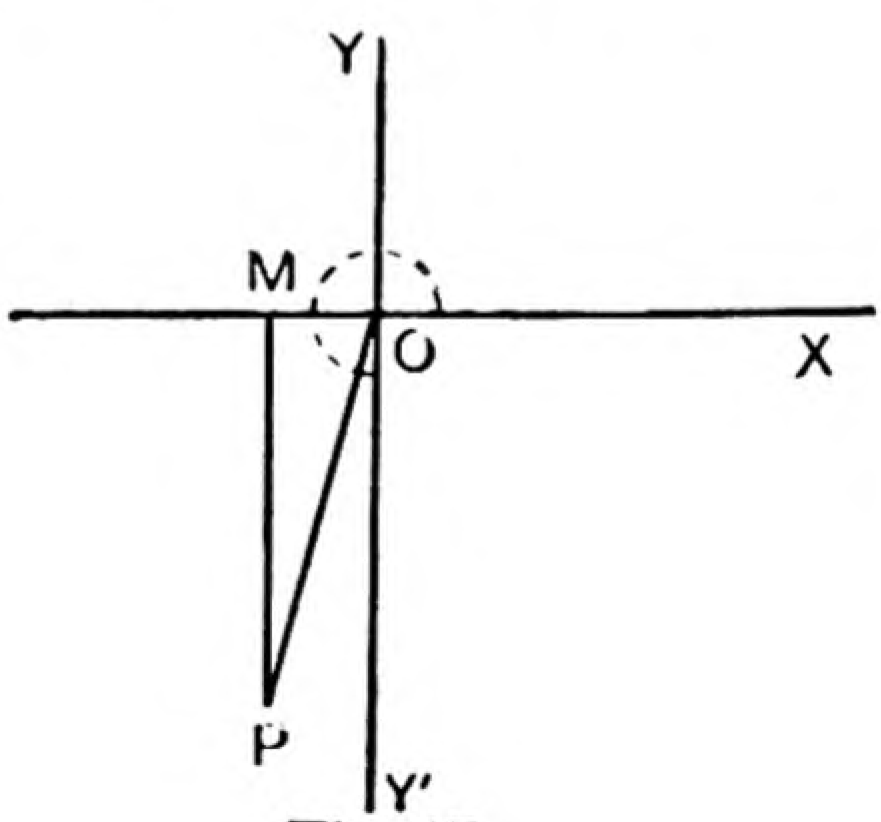


Fig. 67.

In the following proofs, the angle A is supposed to be traced out by a line revolving from the position **OX** to the position **OP**, and the trigonometric functions are defined by means of the auxiliary triangle **OMP**. There is no restriction as to the size of the angle A.

74. The positive powers of the trigonometric functions are written thus: $\sin^2 A$ denotes the square of $\sin A$, and is therefore an abbreviation for $(\sin A)^2$. Similarly, $\tan^3 A$ means the cube of $\tan A$; and so on.*

75. To prove that

$$\tan A = \frac{\sin A}{\cos A} \quad\text{......................}\quad (30)$$

$$\cot A = \frac{\cos A}{\sin A} \quad\text{......................}\quad (31)$$

Fig. 68.

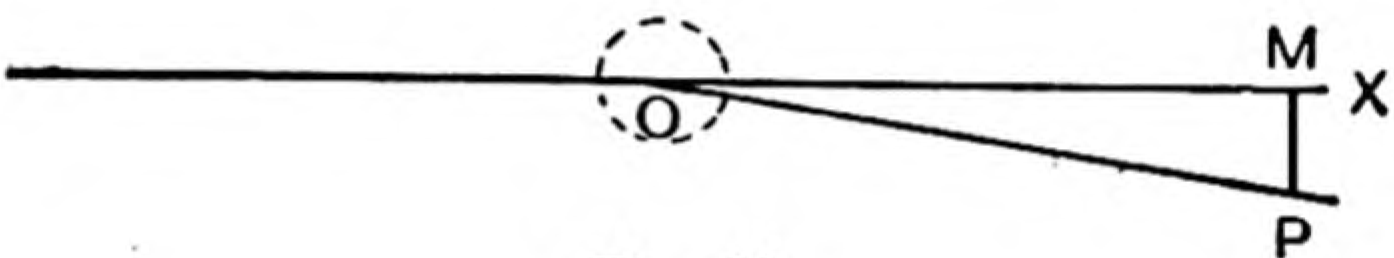


Fig. 69.

These follow at once from the definitions for

$$\sin A \div \cos A = \frac{\mathsf{MP}}{\mathsf{OP}} \div \frac{\mathsf{OM}}{\mathsf{OP}} \left(= \frac{\mathsf{MP}}{\mathsf{OP}} \times \frac{\mathsf{OP}}{\mathsf{OM}} \right) = \frac{\mathsf{MP}}{\mathsf{OM}} = \tan A;$$

$$\cos A \div \sin A = \frac{\mathsf{OM}}{\mathsf{OP}} \div \frac{\mathsf{MP}}{\mathsf{OP}} \left(= \frac{\mathsf{OM}}{\mathsf{OP}} \times \frac{\mathsf{OP}}{\mathsf{MP}} \right) = \frac{\mathsf{OM}}{\mathsf{MP}} = \cot A.$$

76. To prove the relations—

$$\sin^2 A + \cos^2 A = 1 \quad\text{..................}\quad (32)$$

$$\sec^2 A = 1 + \tan^2 A \quad\text{..................}\quad (33)$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A \quad\text{..................}\quad (34)$$

* According to the theory of indices in Algebra, $(\sin A)^{-1}$, $(\cos A)^{-1}$, and $(\tan A)^{-1}$ mean the reciprocals of $\sin A$, $\cos A$, and $\tan A$, respectively; that is, $\operatorname{cosec} A$, $\sec A$, $\cot A$. But the present notation must *not* be used *except for positive powers*, as an entirely different meaning has been assigned to such forms as $\sin^{-1} A$ (see Chap. IX.).

By Euclid I. 47, with the notation of Figs. 66-69,

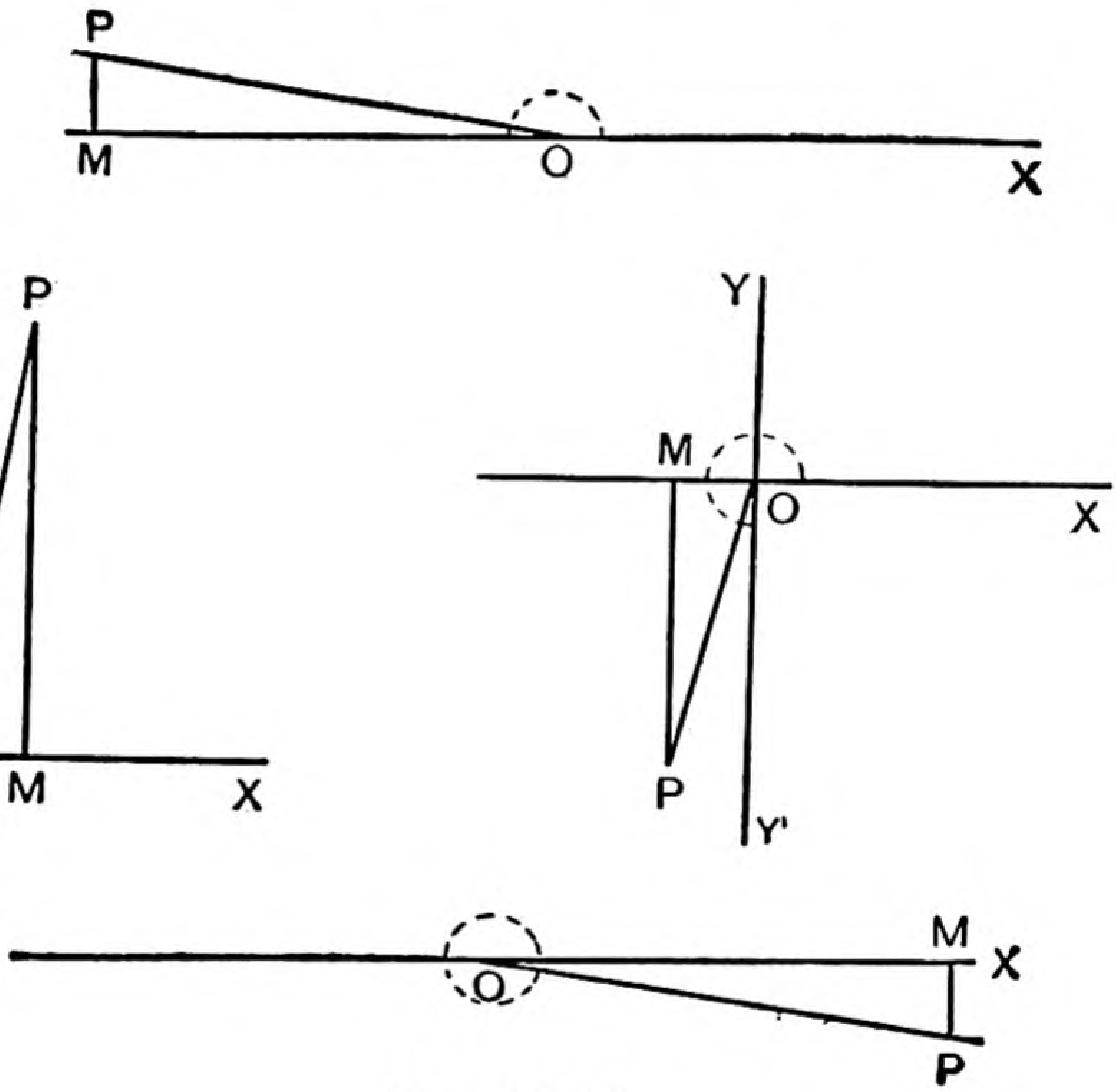
$$\text{sq. on } \mathbf{MP} + \text{sq. on } \mathbf{OM} = \text{sq. on } \mathbf{OP},$$

that is, $$\mathbf{MP}^2 + \mathbf{OM}^2 = \mathbf{OP}^2.$$

Dividing this identity by $\mathbf{OP}^2$, we have

$$\left(\frac{\mathbf{MP}}{\mathbf{OP}}\right)^2 + \left(\frac{\mathbf{OM}}{\mathbf{OP}}\right)^2 = 1,$$

that is, $$\sin^2 A + \cos^2 A = 1.$$



Figs. 66–69.

Again dividing the same identity by $\mathbf{OM}^2$ and $\mathbf{MP}^2$, respectively we have

$$\left(\frac{\mathbf{MP}}{\mathbf{OM}}\right)^2 + \left(\frac{\mathbf{OM}}{\mathbf{OM}}\right)^2 = \left(\frac{\mathbf{OP}}{\mathbf{OM}}\right)^2, \text{ that is, } \tan^2 A + 1 = \sec^2 A;$$

$$\left(\frac{\mathbf{MP}}{\mathbf{MP}}\right)^2 + \left(\frac{\mathbf{OM}}{\mathbf{MP}}\right)^2 = \left(\frac{\mathbf{OP}}{\mathbf{MP}}\right)^2, \text{ that is, } 1 + \cot^2 A = \operatorname{cosec}^2 A;$$

as were to be proved.

77. Summary.—We thus have the following eight very important relations *which may be regarded as fundamental formulae, and which should be remembered*:—

$$\text{cosec} = \frac{1}{\sin} \text{ .... } (a); \qquad \sec = \frac{1}{\cos} \text{ .... } (b);$$

$$\cot = \frac{1}{\tan} \text{ .... } (c); \qquad \tan = \frac{\sin}{\cos} \text{ .... } (d);$$

$$\cot = \frac{\cos}{\sin} \text{ .... } (e); \qquad \sin^2 + \cos^2 = 1 \text{ ...... } (f);$$

$$\sec^2 = 1 + \tan^2 \text{ ...... } (g); \qquad \text{cosec}^2 = 1 + \cot^2 \text{ ... } (h).$$

These formulae are **identities**, that is, they are satisfied by the functions of *any angle whatever*. The reader will have little difficulty in verifying that the values found for particular angles in the last chapter satisfy, *e.g.* such relations as

$$\tan 30° = \frac{\sin 30°}{\cos 30°}, \quad \sin^2 45° + \cos^2 45° = 1, \quad \sec^2 60° = 1 + \tan^2 60°,$$

and so on.

From these eight formulae, any trigonometric function of an angle can be expressed in terms of any other function as illustrated in the examples given below.

The formulae are not all independent, for, if one of the six functions of A be given, *five* simultaneous equations will suffice to determine the five (unknown) values of the remaining functions, and three out of the eight formulae are therefore superfluous (as will be proved more fully in Ex. 3 below).

But it is convenient to retain them all, and to use whichever may happen to be most convenient for the solution of any particular problem.

Ex. 1. If $\cos\theta = \frac{3}{5}$, find the other ratios.

From $$\sin^2\theta + \cos^2\theta = 1.$$

we have $$\sin^2\theta = 1 - \cos^2\theta = 1 - \tfrac{9}{25} = \tfrac{16}{25};$$

$$\therefore \quad \sin\theta = \pm\tfrac{4}{5}.$$

If θ is an angle in the first quadrant, then $\sin\theta$ must be taken with the positive sign, and we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \tfrac{4}{3}, \qquad \text{cosec}\,\theta = \frac{1}{\sin\theta} = \tfrac{5}{4},$$

$$\sec\theta = \frac{1}{\cos\theta} = \tfrac{5}{3}, \qquad \cot\theta = \frac{1}{\tan\theta} = \tfrac{3}{4}.$$

If $\sin\theta$ be taken negative, we have, similarly,

$$\tan\theta = -\tfrac{4}{3}, \quad \text{cosec}\,\theta = -\tfrac{5}{4}, \quad \sec\theta = \tfrac{5}{3}, \quad \cot\theta = -\tfrac{3}{4}.$$

Ex. 2. To express $\sin A$ in terms of $\tan A$, and *vice versa*.*

Substituting $\cot A = \dfrac{1}{\tan A}$ and $\operatorname{cosec} A = \dfrac{1}{\sin A}$

in $$1+\cot^2 A = \operatorname{cosec}^2 A,$$

we have $$1+\frac{1}{\tan^2 A} = \frac{1}{\sin^2 A};$$

$$\therefore \frac{1}{\sin^2 A} = \frac{1+\tan^2 A}{\tan^2 A};$$

$$\therefore \sin^2 A = \frac{\tan^2 A}{1+\tan^2 A} \text{ and } \sin A = \frac{\tan A}{\pm\sqrt{(1+\tan^2 A)}}.$$

Also $$\frac{1}{\tan^2 A} = \frac{1}{\sin^2 A} - 1 = \frac{1-\sin^2 A}{\sin^2 A};$$

$$\therefore \tan^2 A = \frac{\sin^2 A}{1-\sin^2 A} \text{ and } \tan A = \frac{\sin A}{\pm\sqrt{(1-\sin^2 A)}}.$$

Otherwise thus:— $\tan A = \sin A \div \cos A$;

but $\sin^2 A + \cos^2 A = 1$; $\therefore$ $\cos A = \pm\sqrt{(1-\sin^2 A)}$;

$$\therefore \tan A = \frac{\sin A}{\pm\sqrt{(1-\sin^2 A)}}, \text{ as before.}$$

Ex. 3. To deduce the formulae

$$\cot A = \cos A \div \sin A, \quad \sec^2 A = 1+\tan^2 A, \quad \operatorname{cosec}^2 A = 1+\cot^2 A$$

from the other five identities of § 77.

By (*c*), $$\cot A = \frac{1}{\tan A} = \frac{1}{\sin A/\cos A} \quad [\text{by } (d)]$$

$$= \frac{\cos A}{\sin A}.$$

Again dividing the identity $\sin^2 A+\cos^2 A = 1$ by $\cos^2 A$ and $\sin^2 A$, respectively, we have

$$\left(\frac{\sin A}{\cos A}\right)^2 + 1 = \left(\frac{1}{\cos A}\right)^2 \text{ and } 1 + \left(\frac{\cos A}{\sin A}\right)^2 = \left(\frac{1}{\sin A}\right)^2,$$

whence, by (*a*), (*b*), (*d*), and (*e*), which has just been proved,

$$\tan^2 A+1 = \sec^2 A \quad \text{and} \quad 1+\cot^2 A = \operatorname{cosec}^2 A,$$

as required.

78. To express any trigonometric function in terms of any other, the method of the following examples is very convenient and short:—

* Another method of obtaining the present results will be suggested by § 78.

Ex 1. To express the trigonometric functions of an acute angle in terms of the sine.

Let the given sine $= s$.

Then, in the fundamental triangle **ABC**, we have

$$\sin\theta = \mathbf{BC} \div \mathbf{AC} = s.$$

Hence, if we make the denominator $\mathbf{AC} = 1$, then the numerator $\mathbf{BC} = s$, and, by Euclid I. 47,

$$\mathbf{AB} = \sqrt{(1-s^2)}.$$

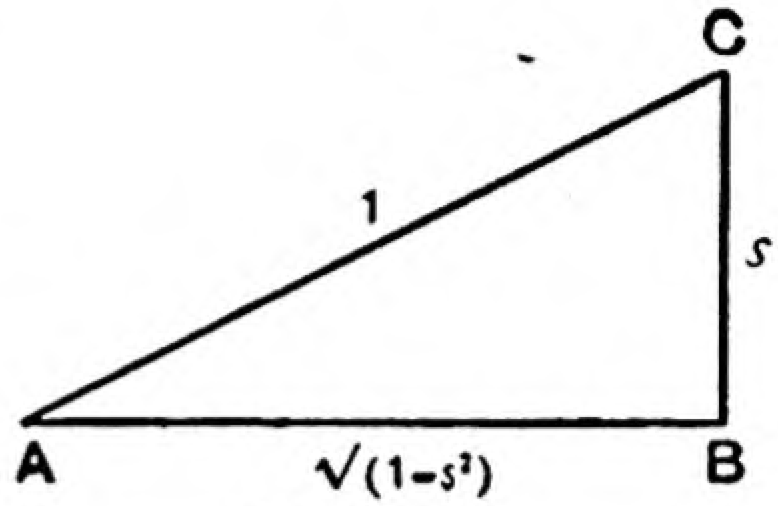


Fig. 70.

The other trigonometric functions may now be read off the figure, thus—

$$\sin\theta = s, \qquad \operatorname{cosec}\theta = \frac{1}{s} = \frac{1}{\sin\theta},$$

$$\cos\theta = \frac{\sqrt{(1-s^2)}}{1} = \sqrt{(1-\sin^2\theta)}, \ \sec\theta = \frac{1}{\sqrt{(1-s^2)}} = \frac{1}{\sqrt{(1-\sin^2\theta)}},$$

$$\tan\theta = \frac{s}{\sqrt{(1-s^2)}} = \frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}, \ \cot\theta = \frac{\sqrt{(1-s^2)}}{s} = \frac{\sqrt{(1-\sin^2\theta)}}{\sin\theta}.$$

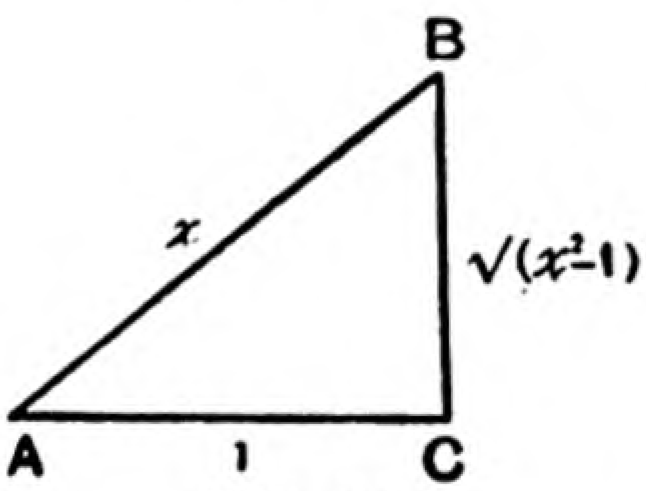


Fig. 71.

Ex. 2. Express all the rest in terms of the secant.

Let $\sec\theta = x$.

Then, in the fundamental triangle **CAB**, $\sec \mathbf{CAB} = \mathbf{AB} \div \mathbf{AC}$.

Hence, if we take the measure of the base **AC** equal to unity,

$$\text{hyp. } \mathbf{AB} = x,$$

and $\quad$ perp. $\mathbf{CB} = \pm\sqrt{(x^2-1)}$. (Euc. I. 47.)

Therefore, by the definitions of the trigonometric functions,

$$\sin\theta = \frac{\pm\sqrt{(x^2-1)}}{x}, \qquad \cos\theta = \frac{1}{x}, \qquad \tan\theta = \pm\sqrt{(x^2-1)},$$

$$\cot\theta = \frac{1}{\pm\sqrt{(x^2-1)}}, \qquad \sec\theta = x, \quad \operatorname{cosec}\theta = \frac{x}{\pm\sqrt{(x^2-1)}},$$

The results must not be left in this form; we must now write $\sec\theta$ for x, and we obtain

$$\sin\theta = \frac{\pm\sqrt{(\sec^2\theta-1)}}{\sec\theta}, \qquad \cos\theta = \frac{1}{\sec\theta}, \qquad \tan\theta = \pm\sqrt{(\sec^2\theta-1)},$$

$$\cot\theta = \frac{1}{\pm\sqrt{(\sec^2\theta-1)}}, \qquad \operatorname{cosec}\theta = \frac{\sec\theta}{\pm\sqrt{(\sec^2\theta-1)}}.$$

These results agree with the Table on page 80.

Ex. 3. To express all the functions in terms of the versed sine. (§ 37) If the versin be given, $= v$, then

$$1-\cos\theta = v;$$

$$\therefore \quad \cos\theta, \text{ or base} \div \text{hyp.} = 1-v.$$

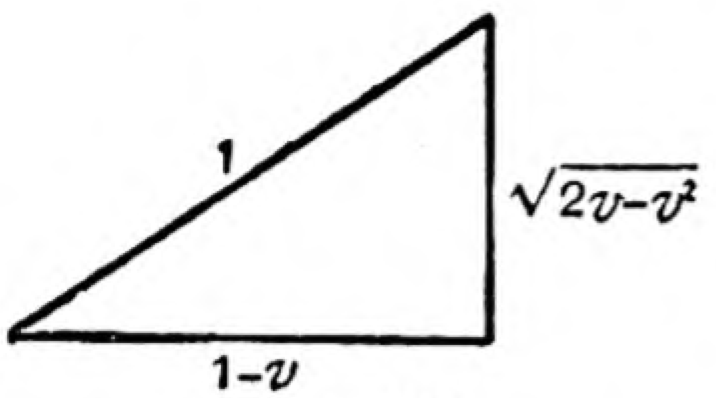


Fig. 72.

Hence we must take the measure of the hypotenuse equal to 1, and the measures of the base and perpendicular will then be

$$1-v \text{ and } \sqrt{\{1-(1-v)^2\}}, \text{ i.e. } \sqrt{(2v-v^2)},$$

respectively, as in Fig. 72.

The other trigonometric functions may now be read off the figure, thus—

$$\sin\theta = \sqrt{(2v-v^2)}, \quad \cos\theta = 1-v, \quad \tan\theta = \frac{\sqrt{(2v-v^2)}}{1-v}, \text{ etc.}$$

ILLUSTRATIVE EXERCISES.

1. Express all the trigonometric functions of an angle in terms of (i) the cosine, (ii) the tangent, (iii) the cotangent, (iv) the cosecant, (v) the coversed sine.

2. In Ex. 3, write down the values of the secant, cosecant, tangent, and coversed sine in terms of the versed sine.

79. A slight modification of the above method is useful when the given trigonometric function is a given fraction.

Ex. 1. Given $\operatorname{cosec}\theta = \frac{25}{7}$, find the other ratios.

In the fundamental triangle,

$$\operatorname{cosec}\theta = \frac{\text{hypot.}}{\text{perp.}} = \frac{25}{7}.$$

Take hypotenuse $= 25$; then perpendicular side $= 7$,

and $\quad (\text{base})^2 = 25^2 - 7^2$

$$= 576;$$

$\therefore \quad$ base $= 24$;

$\therefore \quad \sin\theta = \frac{7}{25}$,

$\cos\theta = \frac{24}{25}$,

$\tan\theta = \frac{7}{24}$, etc.

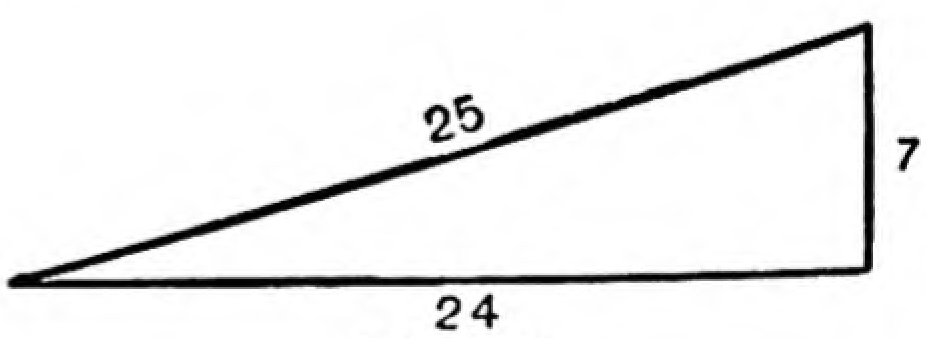


Fig. 73.

Ex. 2. If $\cot\theta = a/b$, to find $\operatorname{cosec}\theta$.

In the fundamental triangle,

$$\cot\theta = \frac{\text{base}}{\text{perp.}} = \frac{a}{b}.$$

Hence, if we take the side adjacent to the angle to be a, the perpendicular side will be b, and, by Euclid I. 47, the

$$\text{hyp.} = \sqrt{(a^2+b^2)};$$

$$\therefore \quad \operatorname{cosec} \theta = \frac{\text{hyp.}}{\text{perp.}} = \frac{\sqrt{(a^2+b^2)}}{b}.$$

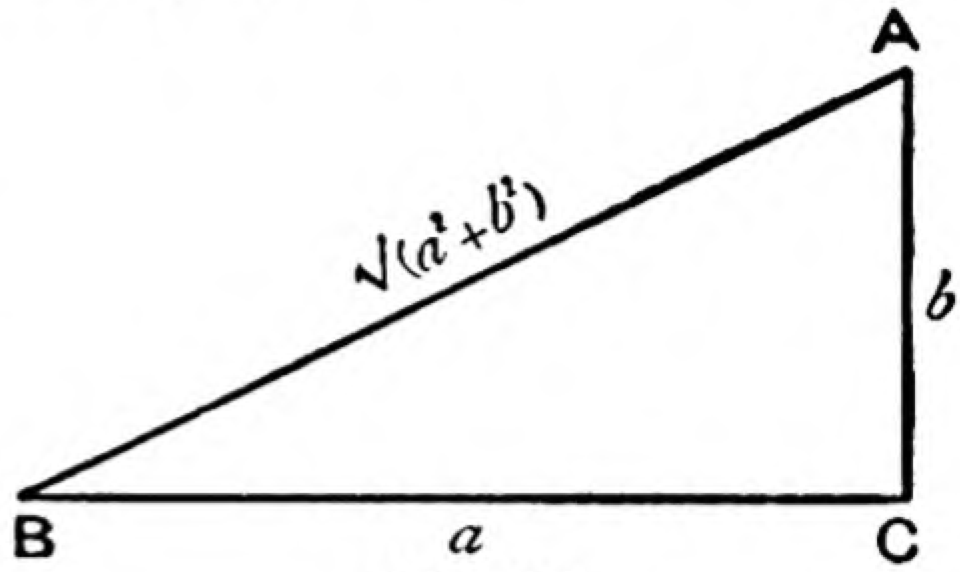


Fig. 74.

80. **Table of results.**—By expressing different functions of an angle in terms of the other functions, we shall obtain the results given on page 80.

The student should on no account attempt to remember this table, but, on the other hand, the present subject should not be left until a table has been reproduced by ruling a square of paper with the necessary spaces, and filling in the blanks, working out each separate entry by the method illustrated in the examples of § 78.

When this has been completed, reference should be made to the present table to see that the results agree.

81. **On Ambiguities of Sign.**—The following examples illustrate very important principles.

Ex. 1. The sine of an angle in the second quadrant is $\frac{3}{5}$; find its remaining trigonometrical functions.

To construct the required angle, proceed as in § 42.

Use Fig. 31 and draw the circle **ABQ** of radius 5.

Draw the perpendicular diameters **AO, BO.** Mark off **OL** of length 3, above **O.**

Draw **PLQ** through **L** parallel to **AO,** cutting the circle in **P** and **Q.**

Join **OQ,** and draw **QN** perpendicular to **AO,**

Then **AOQ** is the required angle.

For $$\sin \mathbf{AOQ} = \frac{\mathbf{NQ}}{\mathbf{OQ}} = \frac{\mathbf{OL}}{\mathbf{OQ}} = \tfrac{3}{5}.$$

Now $$\mathbf{ON}^2 = \mathbf{OQ}^2 - \mathbf{QN}^2 = 5^2 - 3^2 = 16;$$

$$\therefore \quad \mathbf{ON} = \pm 4.$$

But since **ON** *lies to the left of* **O**, *its sign is negative.*

Thus $$\mathbf{ON} = -4.$$

$$\therefore \cos \mathbf{AOQ} = \frac{\mathbf{ON}}{\mathbf{OQ}} = \frac{-4}{5} = -\tfrac{4}{5};$$

$$\tan \mathbf{AOQ} = \frac{\mathbf{NQ}}{\mathbf{ON}} = \frac{3}{-4} = -\tfrac{3}{4}, \text{ etc.}$$

	Sine.	Cosine.	Tangent.	Cotangent.	Secant.	Cosecant.	Versed Sine.
$\sin\theta=$	$\sin\theta$	$\sqrt{(1-\cos^2\theta)}$	$\frac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$	$\frac{1}{\sqrt{(1+\cot^2\theta)}}$	$\frac{\sqrt{(\sec^2\theta-1)}}{\text{cec}\ \theta}$	$\frac{1}{\operatorname{cosec}\theta}$	$\sqrt{(2\operatorname{vers}\theta-\operatorname{vers}^2\theta)}$
$\cos\theta=$	$\sqrt{(1-\sin^2\theta)}$	$\cos\theta$	$\frac{1}{\sqrt{(1+\tan^2\theta)}}$	$\frac{\cot\theta}{\sqrt{(1+\cot^2\theta)}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{(\operatorname{cosec}^2\theta-1)}}{\operatorname{cosec}\theta}$	$1-\operatorname{vers}\theta$
$\tan\theta=$	$\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}$	$\frac{\sqrt{(1-\cos^2\theta)}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{(\sec^2\theta-1)}$	$\frac{1}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$	$\frac{\sqrt{(2\operatorname{vers}\theta-\operatorname{vers}^2\theta)}}{1-\operatorname{vers}\theta}$
$\cot\theta=$	$\frac{\sqrt{(1-\sin^2\theta)}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{(1-\cos^2\theta)}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{(\sec^2\theta-1)}}$	$\sqrt{(\operatorname{cosec}^2\theta-1)}$	$\frac{1-\operatorname{vers}\theta}{\sqrt{(2\operatorname{vers}\theta-\operatorname{vers}^2\theta)}}$
$\sec\theta=$	$\frac{1}{\sqrt{(1-\sin^2\theta)}}$	$\frac{1}{\cos\theta}$	$\sqrt{(1+\tan^2\theta)}$	$\frac{\sqrt{(1+\cot^2\theta)}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$	$\frac{1}{1-\operatorname{vers}\theta}$
$\operatorname{cosec}\theta=$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{(1-\cos^2\theta)}}$	$\frac{\sqrt{(1+\tan^2\theta)}}{\tan\theta}$	$\sqrt{(1+\cot^2\theta)}$	$\frac{\sec\theta}{\sqrt{(\sec^2\theta-1)}}$	$\operatorname{cosec}\theta$	$\frac{1}{\sqrt{(2\operatorname{vers}\theta-\operatorname{vers}^2\theta)}}$
$\operatorname{vers}\theta=$	$1-\sqrt{(1-\sin^2\theta)}$	$1-\cos\theta$	$1-\frac{1}{\sqrt{(1+\tan^2\theta)}}$	$1-\frac{\cot\theta}{\sqrt{(1+\cot^2\theta)}}$	$1-\frac{1}{\sec\theta}$	$1-\frac{\sqrt{(\operatorname{cosec}^2\theta-1)}}{\operatorname{cosec}\theta}$	$\operatorname{vers}\theta$

Ex. 2. Find the trigonometrical functions of each of the two angles whose cosine is $-\frac{\sqrt{3}}{2}$, showing clearly to which angle each value refers.

To construct the required angles, proceed as in § 43. Use Fig. 34; make $\mathbf{OA} = 2$, $\mathbf{OM} = -\sqrt{3}$, $\mathbf{PMQ}$ perpendicular to $\mathbf{AM}$. Then the two angles, whose cosine is $-\frac{\sqrt{3}}{2}$, are $\mathbf{AOP}$ and $\mathbf{AOQ}$,

Now $$\mathbf{MP}^2 = \mathbf{OP}^2 - \mathbf{OM}^2 = 4-3 = 1;$$
$$\therefore \quad \mathbf{MP} = \pm 1.$$

Similarly $$\mathbf{MQ} = \pm 1.$$

But the sign of $\mathbf{MP}$ *is positive since it is drawn upwards, and of* $\mathbf{MQ}$ *is negative since it is drawn downwards.*

Thus $$\mathbf{MP} = +1; \quad \mathbf{MQ} = -1.$$

Thus $$\sin \mathbf{AOP} = \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{+1}{2} = +\tfrac{1}{2};$$

$$\tan \mathbf{AOP} = \frac{\mathbf{MP}}{\mathbf{OM}} = \frac{+1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}; \text{ etc.}$$

$$\sin \mathbf{AOQ} = \frac{\mathbf{MQ}}{\mathbf{OQ}} = \frac{-1}{2} = -\tfrac{1}{2};$$

$$\tan \mathbf{AOQ} = \frac{\mathbf{MQ}}{\mathbf{OM}} = \frac{-1}{-\sqrt{3}} = +\frac{1}{\sqrt{3}}; \text{ etc.}$$

Ex. 3. Discuss the ambiguous signs in the formulae

$$\cos A = \pm\sqrt{1-\sin^2 A},$$

$$\sec A = \pm\frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

If A falls in the first quadrant, all the trigonometrical ratios are positive; hence

$$\cos A = +\sqrt{1-\sin^2 A}\cdot$$

$$\sec A = +\frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

If A falls in the second quadrant, $\cos A$ is negative (§ 38); thus

$$\cos A = -\sqrt{1-\sin^2 A}.$$

Also since *sec A and cot A are both negative* (§ 38), the second equation must be written

$$\sec A = +\frac{\sqrt{1+\cot^2 A}}{\cot A},$$

for both sides of the equation now represent negative quantities.

If A falls in the third quadrant, $\cos A$ is negative; thus

$$\cos A = -\sqrt{1-\sin^2 A}.$$

Also since *sec A is negative and cot A is positive* (§ 38), the second equation must be written

$$\sec A = -\frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

If A falls in the fourth quandrant, $\cos A$ is positive; thus

$$\cos A = +\sqrt{1-\sin^2 A}.$$

Also since *sec A is positive and cot A is negative,* the second equation must be written

$$\sec A = -\frac{\sqrt{1+\cot^2 A}}{\cot A},$$

where both sides of the equation are now positive quantities.

82. **Difference between an identity and an equation.**—The eight formulae of § 77 have already been referred to as *identities.* Before proceeding further, it will be convenient to recapitulate the following definitions, which apply to Trigonometry as well as to Algebra:—

DEF.—Any quantity which is capable of assuming different values in a mathematical expression is called a **variable.** But, when any definite numerical value is assigned to the quantity, it no longer remains a variable, but is called a **constant.**

Thus, in the relation $\sec^2\theta = 1+\tan^2\theta$, the angle θ is a *variable,* and $\sec\theta$ and $\tan\theta$ are consequently also variables, since their values change when θ changes.

But, if we put $\theta = \frac{1}{4}\pi$ (radians), θ becomes a *constant,* and so do $\sec\theta$ and $\tan\theta$, for $\sec\theta = \sqrt{2}$ and $\tan\theta = 1$.

DEF.—If two expressions are equal for *all* values of the variables involved in them, the statement of their equality is called an **identity.** But, if the expressions are only equal when the variables assume certain definite values, the statement of their equality is called an **equation,** and the process of finding what values of the variables make the two expressions equal is called **solving the equation.** The values themselves are called **solutions,** or **roots,** of the equation.

Thus the statement that $\operatorname{cosec}^2\theta = 1+\cot^2\theta$ is an *identity,* because it is true whatever be the value of θ.

But $\sin\theta = \cos\theta$ is an *equation*, not an identity. For, on dividing both side by $\cos\theta$, it becomes $\tan\theta = 1$; now this is satisfied when $\theta = \frac{1}{4}\pi$ (radians) $= 45°$, but it does not hold good when θ is any other angle in the first quadrant. Hence $\theta = \frac{1}{4}\pi$ is a solution or *root* of the equation $\sin\theta = \cos\theta$.

83. **Trigonometric identities.**—By combining the eight fundamental formulae (*a*) to (*h*) in various ways, innumerable other more or less complicated identities can be built up. Conversely, when two given expressions involving the trigonometric functions of an angle have to be proved equal, this can always be effected by suitably transforming one or both of the expressions by means of the same formulae, and it is usually advisable to observe some such rules as the following:—

(1) Express cosecants and secants in terms of sines and cosines.

(2) Treat the more complicated side of the identity first, expressing all the functions it involves in terms of one of them if this can be done without introducing radicals. If not, express all the functions in terms of two of them (usually the sine and cosine), and simplify as far as possible, avoiding radicals (*e.g.* using the identity $\sin^2 + \cos^2 = 1$, where sines or cosines occur *squared*).

(3) If the simplified expression thus obtained is not easily transformed into the other side of the identity, take the latter and transform it in like manner, expressing it in terms of the same function or functions as the first side.

The two sides will now be found to be identically equal.

Sometimes the work may be shortened by artifices which *should naturally suggest themselves* to the student; thus, if $1+\tan^2 A$ occurs in an identity, we usually replace it by $\sec^2 A$; on the other hand, $\sec^2 A - 1$ may be replaced by $\tan^2 A$.

Ex. 1. Prove that $\operatorname{cosec}^2\theta + \sec^2\theta = \operatorname{cosec}^2\theta\sec^2\theta$.

By (1) and (2),

$$\operatorname{cosec}^2\theta + \sec^2\theta = \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \operatorname{cosec}^2\theta . \sec^2\theta.$$

Ex. 2. Prove that $\tan^2\theta-\sin^2\theta=\sin^4\theta\sec^2\theta$.

$$\tan^2\theta-\sin^2\theta=\frac{\sin^2\theta}{\cos^2\theta}-\sin^2\theta=\frac{\sin^2\theta-\sin^2\theta\cos^2\theta}{\cos^2\theta}$$

$$=\frac{\sin^2\theta}{\cos^2\theta}(1-\cos^2\theta)=\frac{\sin^2\theta}{\cos^2\theta}\sin^2\theta=\frac{\sin^4\theta}{\cos^2\theta}$$

$$=\sin^4\theta\sec^2\theta.$$

Ex. 3. Prove that $(\sec A-\operatorname{cosec} A)(1+\cot A+\tan A)$

$$=\frac{\sec^2 A}{\operatorname{cosec} A}-\frac{\operatorname{cosec}^2 A}{\sec A}.$$

$(\sec A-\operatorname{cosec} A)(1+\cot A+\tan A)$

$$=\left(\frac{1}{\cos A}-\frac{1}{\sin A}\right)\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)$$

$$=\frac{\sin A-\cos A}{\sin A\cos A}\cdot\left(1+\frac{\cos^2 A+\sin^2 A}{\sin A\cos A}\right)$$

$$=\frac{(\sin A-\cos A)(\sin A\cos A+1)}{\sin^2 A\cos^2 A}\quad(\because\quad\cos^2 A+\sin^2 A=1)$$

$$=\frac{\sin^2 A\cos A-\cos^2 A\sin A+\sin A-\cos A}{\sin^2 A\cos^2 A}$$

$$=\frac{\sin A(1-\cos^2 A)-\cos A(1-\sin^2 A)}{\sin^2 A\cos^2 A}=\frac{\sin A\sin^2 A-\cos A\cos^2 A}{\sin^2 A\cos^2 A}$$

$$=\frac{\sin A}{\cos^2 A}-\frac{\cos A}{\sin^2 A}.$$

Again,

$$\frac{\sec^2 A}{\operatorname{cosec} A}-\frac{\operatorname{cosec}^2 A}{\sec A}=\frac{1/\cos^2 A}{1/\sin A}-\frac{1/\sin^2 A}{1/\cos A}=\frac{\sin A}{\cos^2 A}-\frac{\cos A}{\sin^2 A},$$

which proves the identity.

84. Caution.—The plan, so often adopted by students, of *assuming* what they have to prove, and deducing some such result as $1=1$, should be avoided. As a last resource, however, and one only to be used if a direct proof cannot be given, an identity may be proved indirectly *provided that each step of the process be carefully qualified and made to depend on the next step*, as in the following example:—

Ex. To prove that

$$(\sec\theta+\operatorname{cosec}\theta)(\sin\theta+\cos\theta)=\sec\theta\operatorname{cosec}\theta+2.$$

The identity will be true if

$$\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)(\sin\theta+\cos\theta)=\frac{1}{\sin\theta\cos\theta}+2,$$

that is, if $$(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta,$$

that is, if $$\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta = 1+2\sin\theta\cos\theta.$$

But this is true, because $\sin^2\theta+\cos^2\theta = 1$.

Therefore the proposed identity is true.

EXAMPLES VII.

1. Prove that $\sin^2 A+\cos^2 A = 1$, and that $\sin A = \pm\dfrac{\tan A}{\sqrt{1+\tan^2 A}}$.

2. If $\sin a = \frac{12}{13}$, find $\cos a$ and $\tan a$, assuming that a is an acute angle.

3. Given $\cos A = \frac{3}{4}$, show how to construct the angle A, assuming that it is in the first quadrant; and find the sine, tangent, and cotangent of A.

4. Given $\tan B = -\frac{4}{3}$, find the sine, cosine, and cotangent of B.

5. The secant of a certain angle is 2; find all the other functions.

5*a*. Given $\sin A = -\frac{3}{5}$, in which quadrants may A lie? Give the values of $\cos A$ and $\cot A$ in each case.

5*b*. Given $\cos A = -\frac{1}{2}\sqrt{2}$, in which quadrants may A lie? Give the values of $\operatorname{cosec} A$ and $\tan A$ in each case.

5*c*. Given $\tan A = \dfrac{a}{b}$, where a and b represent positive quantities, in which quadrants may A lie? Give the value of $\cos A$ and $\operatorname{cosec} A$ in each case.

5*d*. Given $\operatorname{cosec} A = -\dfrac{l}{m}$, where l and m represent positive quantities, in which quadrants may A lie? Give the value of $\sec A$, and $\tan A$ in each case.

5*e*. Determine which sign to use in the different quadrants, in the following formulae.

(i) $\tan\theta = \pm\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$; (ii) $\operatorname{cosec}\theta = \pm\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$;

(iii) $\cot\theta = \pm\dfrac{1-\operatorname{vers}\theta}{\sqrt{(2\operatorname{vers}\theta-\operatorname{vers}^2\theta)}}$.

PROVE the following identities (6-36):—

6. $\sin\theta\tan\theta = \sec\theta-\cos\theta$.

7. $\cos\theta\cot\theta = \operatorname{cosec}\theta-\sin\theta$.

8. $\cos^4\theta-\sin^4\theta = 1-2\sin^2\theta$.

9. $\tan\theta+\cot\theta = \sec\theta\operatorname{cosec}\theta$.

10. $\operatorname{cosec}^2\theta+\sec^2\theta = \sec^2\theta\operatorname{cosec}^2\theta$.

11. $\dfrac{\operatorname{cosec}\theta}{\sec\theta}+\dfrac{\sec\theta}{\operatorname{cosec}\theta} = \sec\theta\operatorname{cosec}\theta$.

12. $\cot^2\theta-\cos^2\theta = \cos^4\theta\operatorname{cosec}^2\theta$.

13. $\sin^2\theta+\operatorname{vers}^2\theta = 2(1-\cos\theta)$.

14. $\cos^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\sin^2\theta\cos^2\phi+\sin^2\phi\cos^2\theta = 1.$

15. $(\tan\theta+\cot\theta)^2 = \sec^2\theta+\operatorname{cosec}^2\theta.$

16. $\sec^4\theta+\tan^4\theta = 1+2\sec^2\theta\tan^2\theta.$

17. $\sin^2\theta+\cos^4\theta = \cos^2\theta+\sin^4\theta.$

18. $\tan\theta-\cot\theta = (\tan\theta-1)(\cot\theta+1).$

19. $\sin^2\theta\tan^2\theta+\cos^2\theta\cot^2\theta = \tan^2\theta+\cot^2\theta-1.$

20. $\{\sqrt{(\sec\theta+\tan\theta)}+\sqrt{(\sec\theta-\tan\theta)}\}^2 = 2(1+\sec\theta).$

21. $\{\sqrt{(\operatorname{cosec}\theta+\cot\theta)}-\sqrt{(\operatorname{cosec}\theta-\cot\theta)}\}^2 = 2(\operatorname{cosec}\theta-1).$

22. $\cos\theta = \sin\theta\tan^2\theta\cot^3\theta.$

23. $(\sin\theta+\cos\theta)(\tan\theta+\cot\theta) = \sec\theta+\operatorname{cosec}\theta.$

24. $\sec^6\theta-\tan^6\theta = 1+3\tan^2\theta\sec^2\theta.$

25. $\sec^2\theta-\sec^2\phi = \tan^2\theta-\tan^2\phi.$

26. $\sec^4\theta-\sec^2\theta = \tan^4\theta+\tan^2\theta.$

27. $\cos\theta(2\sec\theta+\tan\theta)(\sec\theta-2\tan\theta) = 2\cos\theta-3\tan\theta.$

28. $(\cos A+\sin A)(\operatorname{cosec} A-\sec A) = \cot A-\tan A.$

29. $(1-2\cos^2 B)(\tan B+\cot B) = (\sin B-\cos B)(\sec B+\operatorname{cosec} B).$

30. $\dfrac{\tan A+\tan B}{\cot A+\cot B} = \tan A\tan B.$

31. $\dfrac{\sin C-\sin D}{\cos C+\cos D}+\dfrac{\cos C-\cos D}{\sin C+\sin D} = 0.$

32. $\sin^3 A+\cos^3 A = (1-\sin A\cos A)(\sin A+\cos A).$

33. $\sin^6 A-\cos^6 A$
$= (1-\sin A\cos A)(1+\cos A\sin A)(\sin A-\cos A)(\sin A+\cos A).$

34. $\sin^4\theta+\cos^4\theta = \sin^2\theta(\operatorname{cosec}^2\theta-2\cos^2\theta).$

35. $(\sec\phi-\cos\phi)(\operatorname{cosec}\phi-\sin\phi) = \dfrac{\tan\phi}{1+\tan^2\phi}.$

36. $\cos^6 A+\sin^6 A = 1-3\sin^2 A+3\sin^4 A.$

Simplify the following expressions (37–40):—

37. $1+\dfrac{\tan^2\theta}{1+\sec\theta}.$

38. $\left(\dfrac{1}{\sec^2\theta-\cos^2\theta}+\dfrac{1}{\operatorname{cosec}^2\theta-\sin^2\theta}\right)\times\sin^2\theta\cos^2\theta.$

39. $\sin^2\theta\tan\theta+\cos^2\theta\cot\theta+2\sin\theta\cos\theta.$

40. $3(\sin^4\theta+\cos^4\theta)-2(\sin^6\theta+\cos^6\theta).$

41. At a point on the ground directly opposite the centre of the front of a house, its length subtends an angle double that whose sine is $\frac{4}{5}$, and the height subtends an angle whose cosine is $\frac{3}{5}$. The height of the house is 30 ft. What is its length and how far away is the point of observation?

42. A ladder 26 ft. long, being placed in a street, will just reach a window 10 ft. from the ground on one side; on being turned over without moving the foot so as to be at right angles to its former position it will reach a window on the other side of the street: determine the height of this latter window and the width of the street.

43. Complete the table on page 80 by adding the row and column for the coversed sine.

CHAPTER VIII.

RELATIONS BETWEEN THE FUNCTIONS OF ALLIED ANGLES.

85. DEF.—When two angles together make up a right angle, each is called the **complement** of the other, and the angles are said to be **complementary**.

The complement of A in sexagesimal measure is, therefore, $90°-A$, and the complement of a in circular measure is $\frac{1}{2}\pi-a$.

The angle need not be positive, nor less than a right angle; *e.g.* the complement of $120°$ is $90°-120°$ or $-30°$, and so on.

DEF.—When two angles together make up two right angles, each is called the **supplement** of the other, and the two angles are said to be **supplementary**.

This is the case with the two angles which one straight line makes with another in Euclid I. 13.

The supplement of A in sexagesimal measure is, therefore, $180°-A$, and the supplement of a in circular measure is $\pi-a$.

Ex. The supplement of the complement of an angle exceeds the complement of its supplement by 180°.

For the supplement of the complement of A

$$= 180°-(90°-A) = A+90°,$$

and the complement of the supplement of A

$$= 90°-(180°-A) = A-90°;$$

$$\therefore \text{ difference} = 180°.$$

ILLUSTRATIVE EXERCISES.

1. Write down the complements of $45°$, $-225°$, $196°$, $\frac{1}{3}\pi$, $\frac{3}{2}\pi$, -2π.

2. Write down the supplements of $120°$, $270°$, $-330°$, $\frac{2}{3}\pi$, π, $-\frac{1}{4}\pi$.

3. Find the complement of the supplement and the supplement of the complement of 30°, −30°, $\frac{1}{2}\pi$, $-\frac{3}{4}\pi$.

4. Prove that, if B is the complement of the supplement of A, A is the supplement of the complement of B.

86. To compare the trigonometric functions of two complementary angles.

When the angles are acute, the easiest proof is as follows:—

Let **BAC** be any acute angle $= A$. Take a point **C** on **AC**, and complete the fundamental triangle by drawing **CB** perpendicular to **AB**.

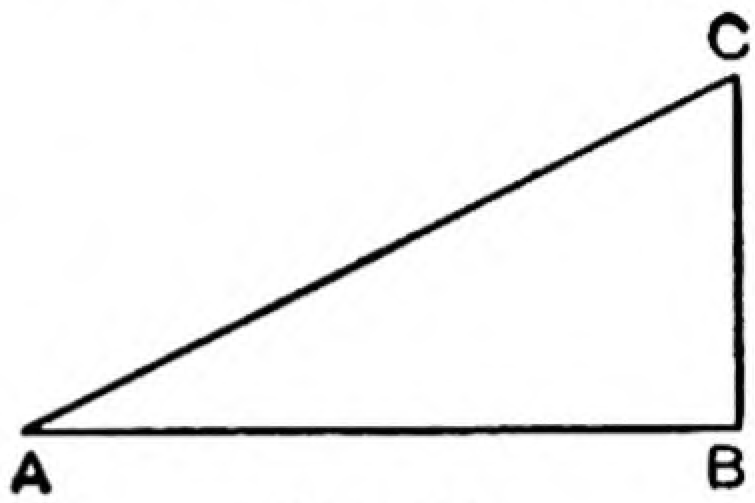


Fig. 75.

Then, since

the three angles of $\triangle$**ABC** = two right angles,

∴ the two angles at **A, C** together = one right angle,

i.e. $\angle C$ is the complement of $\angle A$.

Hence

$$\left.\begin{aligned}
\sin(90°-A) &= \sin \mathbf{ACB} = \frac{\mathbf{BA}}{\mathbf{CA}} = \cos \mathbf{BAC} = \cos A;\\
\text{similarly,}\quad \cos(90°-A) &= \frac{\mathbf{CB}}{\mathbf{CA}} = \sin A \quad \ldots\ldots\ldots\ldots\\
\tan(90°-A) &= \frac{\mathbf{BA}}{\mathbf{CB}} = \cot A \quad \ldots\ldots\ldots\ldots\\
\cot(90°-A) &= \frac{\mathbf{CB}}{\mathbf{BA}} = \tan A \quad \ldots\ldots\ldots\ldots\\
\operatorname{cosec}(90°-A) &= \frac{\mathbf{CA}}{\mathbf{BA}} = \sec A \ldots\ldots\ldots\ldots\ldots\\
\sec(90°-A) &= \frac{\mathbf{CA}}{\mathbf{CB}} = \operatorname{cosec} A \ldots\ldots\ldots\ldots
\end{aligned}\right\} \ldots (35)$$

When reduced to circular measure, these relations become

$$\sin(\tfrac{1}{2}\pi - a) = \cos a, \text{ etc.} \quad \ldots\ldots\ldots\ldots\ldots (35\pi)$$

*87. *When the angles are unrestricted in magnitude*, we proceed thus: Let $\angle$**AOQ** $= A$. Draw **OB** perpendicular to **OA**, and from **OB** in the opposite direction describe the $\angle$**BOP** = **AOQ**. Then $\angle$**AOP** is

negative and equal in *magnitude* to difference between $\angle$ **POB** and $\angle$ **AOB**.

$$\therefore \quad \angle AOP = -(\angle POB - \angle AOB) = \angle AOB - \angle POB$$
$$= 90^\circ - \angle AOQ = 90^\circ - A.$$

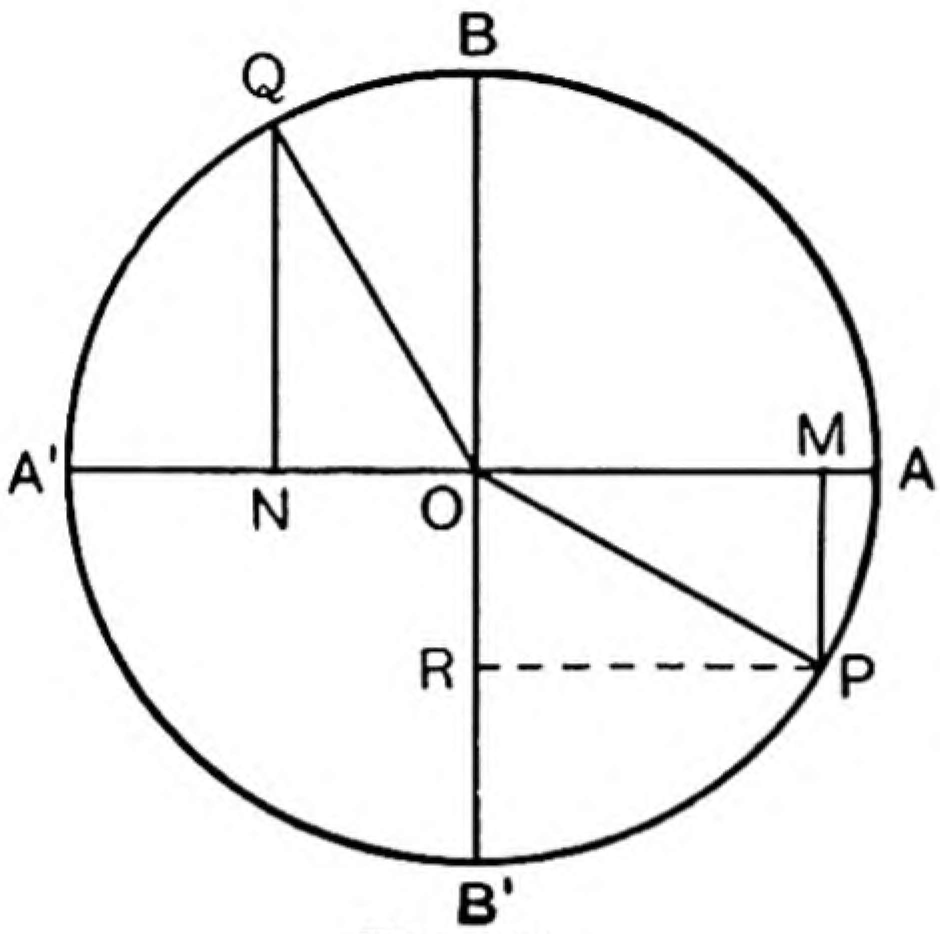


Fig. 76.

Take **OP** = **OQ**, and complete the fundamental triangles **OMP**, **ONQ**, and draw **PR** perpendicular on **OB**.

Then $\triangle$s **OMP**, **ORP** are equal in every respect, and therefore so are **OMP**, **QNO**, but the "perpendicular" or ordinate in one is the "base" or abscissa in the other, and *vice versa*.

Also, **NQ**, **ON** have the same algebraic signs as **OM**, **MP**, respectively;

$$\therefore \quad \sin(90^\circ - A)$$
$$= \frac{MP}{OP} = \frac{OR}{OP} = \frac{ON}{OQ}$$
$$= \cos A, \text{ and so on.}$$

ILLUSTRATIVE EXERCISE.

Draw the figures for the cases when A lies (i) between 180° and 270°, (ii) between 270° and 360°.

88. The results of the previous articles may be remembered by the following rule:—

To express any trigonometric ratio of the complement of an angle as a trigonometric ratio of the angle, either take off or put on the letters **co**.

Thus,
$$\text{cosine of } (90^\circ - A) = \text{sine of } A,$$
$$\sec(90^\circ - A) = \operatorname{cosec} A,$$
$$\operatorname{cotan}(90^\circ - A) = \tan A, \text{ and so on.}$$

This rule is usually stated in more mathematical language, thus:— "*the trigonometric functions of the complement of an angle are equal to the corresponding* co-functions *of the angle, and vice versa.*

89. To compare the trigonometric functions of two supplementary angles.

Let the revolving line trace out the $\angle$ **AOP** = A.

Produce **AO** to **A′**, and from **OA′** in the negative direction, describe
$$\angle A'OQ = A.$$
Then
$$\angle AOQ = 180^\circ - A.$$

Take **OQ** = **OP**, and drop the perpendiculars **PM, QN.**

Then the fundamental triangles **OMP, ONQ** are equal in every respect, but **OM, ON** are measured in opposite directions;

$$\therefore \quad \mathbf{NQ} = \mathbf{MP} \quad \text{and} \quad \mathbf{ON} = -\mathbf{OM};$$

$$\left.\begin{aligned} \therefore \quad \sin(180^\circ - A) &= \frac{\mathbf{NQ}}{\mathbf{OQ}} = \frac{\mathbf{MP}}{\mathbf{OP}} = \sin A \\ \cos(180^\circ - A) &= \frac{\mathbf{ON}}{\mathbf{OQ}} = -\frac{\mathbf{OM}}{\mathbf{OP}} = -\cos A \\ \tan(180^\circ - A) &= \frac{\mathbf{NQ}}{\mathbf{ON}} = \frac{\mathbf{MP}}{-\mathbf{OM}} = -\tan A \end{aligned}\right\} \ldots\ldots\ldots (36)$$

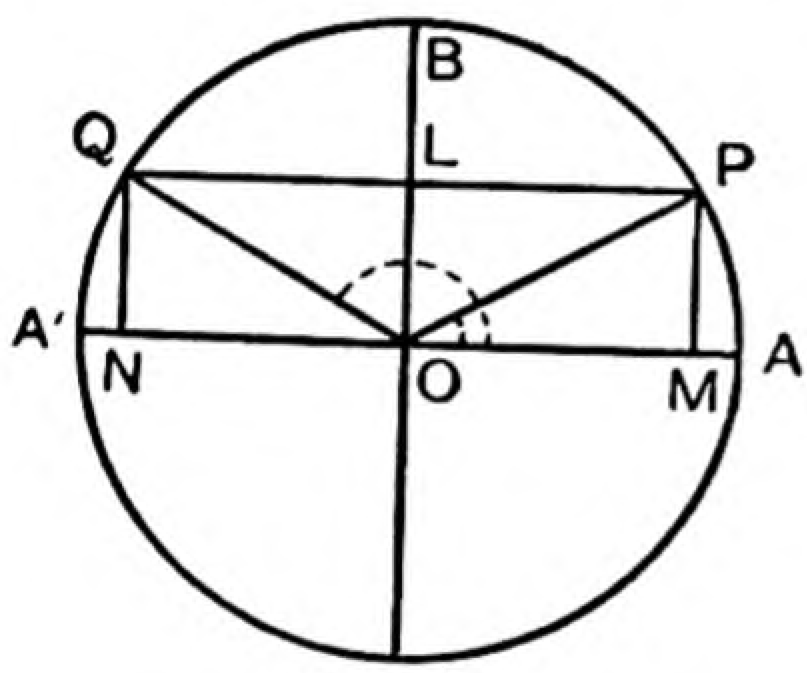


Fig. 77.

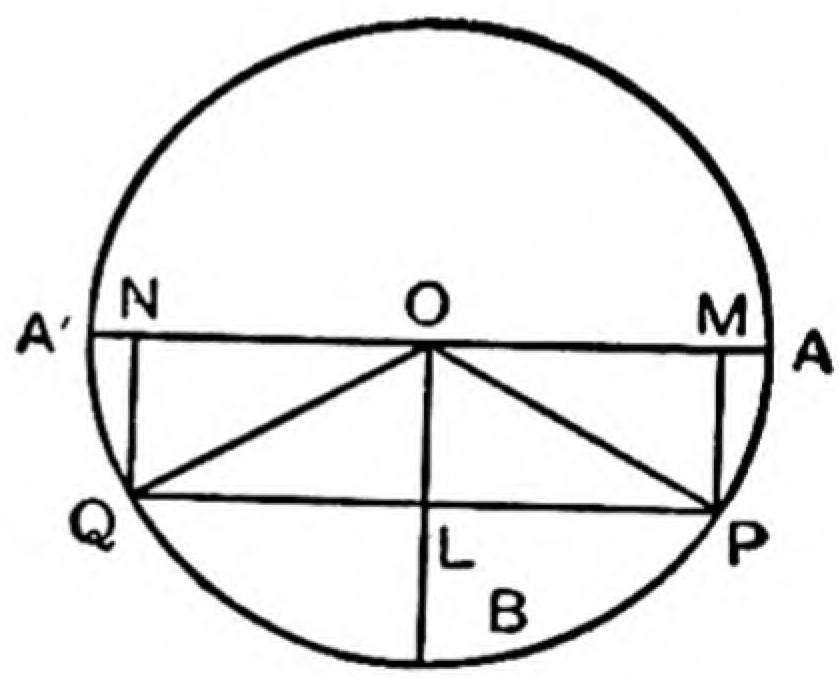


Fig. 78.

Similarly, $\quad \operatorname{cosec}(180^\circ - A) = \operatorname{cosec} A,$

$\sec(180^\circ - A) = -\sec A, \qquad \cot(180^\circ - A) = -\cot A.$

When reduced to circular measure, these relations become

$$\sin(\pi - \alpha) = \sin\alpha, \ \cos(\pi - \alpha) = -\cos\alpha, \text{ etc.} \ldots\ldots (36\pi)$$

If $\angle A$ *is obtuse,* represent it by $\angle$ **AOQ** (Fig. 77); then

$$\angle \mathbf{AOP} = 180^\circ - A,$$

and the same relations follow at once. A similar proof applies to angles of any size. (See Fig. 78.)

Illustrative Exercises.

Draw the figures for the cases (i) when A lies between 180° and 270°, (ii) when A lies between 0 and -90, (iii) when α lies between $\frac{3}{2}\pi$ and 2π.

90. The above results may be summed up as follows:—

The trigonometric ratios of the supplement of an angle have the same *numerical* values as the corresponding ratios of the angle; but, while the *signs* of the *sine* and *cosecant* remain the *same*, those of the remaining four functions are *changed.*

This can be remembered from § 39; for, if the given angle is in the first quadrant and has *all* its functions positive (+), its supplement is in the second quadrant and has only the *sine* and *cosecant* positive (+).

N.B.—The *versed sine* is *not* one of the functions included in this rule, for

$$\text{vers}\,(180°-A) = 1-\cos\,(180°-A) = 1+\cos A = 2-(1-\cos A)$$
$$= 2-\text{vers}\,A.$$

In numerical examples it is instructive to deduce the results independently by drawing a figure instead of applying general formulae; thus—

Ex.. To find sec 150° and tan 150°.

$$150° = 180°-30°.$$

Drawing the figure as in Fig. 77, the triangles **OMP** and **ONQ** have equal bases. Hence the ratios of 150° are numerically equal to those of 30°. Also, since 150° is in the second quadrant, its secant and tangent are negative.

$$\therefore \quad \sec 150° = -\sec 30° = -\tfrac{2}{3}\sqrt{3}, \quad \tan 150° = -\tan 30° = -\tfrac{1}{3}\sqrt{3}.$$

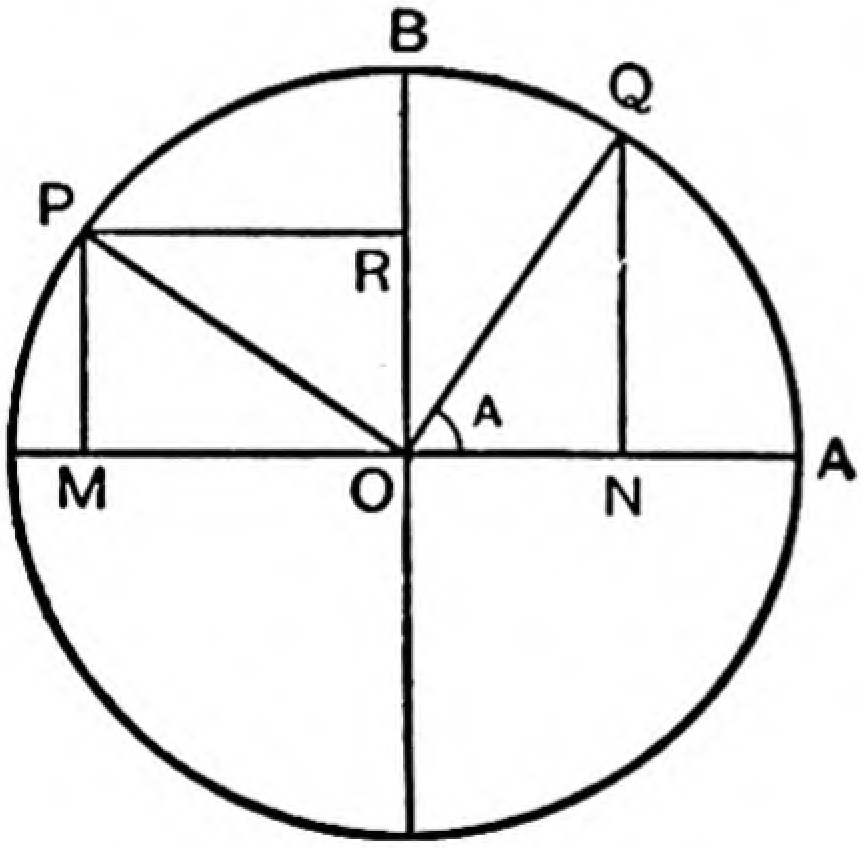


Fig. 79.

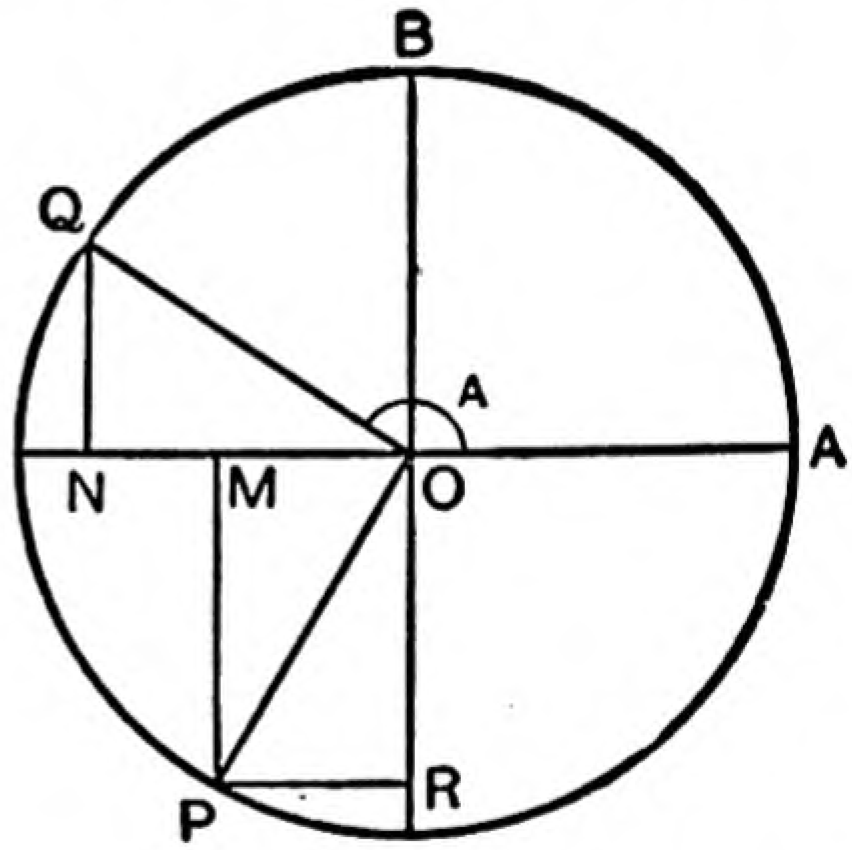


Fig. 80.

91. To compare the trigonometric functions of A and $90°+A$. (*First Method.*)

In Fig. 79, let $\angle$**AOQ** $= A$, $\angle$**AOP** $= 90° + A$.

Then, if **OB** makes with **OA** an angle of 90° in the positive direction, we have $\angle$**BOP** $= A$.

Take **OP** = **OQ**, and draw **PM**, **QN** perpendicular on **OA**, and **PR** on **OB**.

Then $$\angle \mathbf{NOR} = \angle \mathbf{AOB} = 90° = \angle \mathbf{QOP};$$
hence $$\angle \mathbf{QON} = \angle \mathbf{POR}.$$

Hence the triangles **ROP, NOQ** are equal in all respects; and it will be found in every case that **OR** is of the same sign algebraically as **ON**, and **RP** of opposite sign to **NQ**.

$$\left.\begin{aligned}
\therefore\ \sin(90° + A) &= \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{\mathbf{OR}}{\mathbf{OP}} = \frac{\mathbf{ON}}{\mathbf{OQ}} = \cos A \\
\cos(90° + A) &= \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{\mathbf{RP}}{\mathbf{OP}} = -\frac{\mathbf{NQ}}{\mathbf{OQ}} = -\sin A \\
\tan(90° + A) &= \frac{\mathbf{MP}}{\mathbf{OM}} = \frac{\mathbf{OR}}{\mathbf{RP}} = \frac{\mathbf{ON}}{-\mathbf{NQ}} = -\cot A
\end{aligned}\right\} \dots\dots\dots\dots(37)$$

and similarly for the other functions. The same proof applies if the angle A is obtuse. (See Fig. 80.)

Illustrative Exercises.

1. Write down the relations (37) reduced to circular measure.

2. Draw the figures for the cases (i) when A lies between 180° and 270°, (ii) when A lies between 270° and 360°; and in either case prove the formulae connecting the secant, cosecant, and cotangent of $90°+A$ with the functions of A.

92. To compare the trigonometric functions of A and $90 + A$. (*Second Method.*)

Since $$90°+A = 180°-(90°-A),$$
$90°+A$ is the supplement of the complement of A, and its functions may be expressed by means of the two preceding articles.

Thus

$$\left.\begin{aligned}
\sin(90°+A) &= \sin\{180°-(90°-A)\} = \sin(90°-A) = \cos A \\
\cos(90°+A) &= \cos\{180°-(90°-A)\} = -\cos(90°-A) = -\sin A \\
\tan(90°+A) &= \tan\{180°-(90°-A)\} = -\tan(90°-A) = -\cot A
\end{aligned}\right\}$$
$$\dots\dots\dots\dots(37)$$

and so on. The results can be summed up as follows:—

The *trigonometric functions of* $90°+A$ are *numerically* equal to the corresponding co-functions of A, and *vice versa*; but all except the *sin* and *cosec* of $90°+A$ are of opposite *sign*.

As in the last article, we may remember the signs to take by observing that, if A be in the first quadrant, $90°+A$ will be in the second quadrant so that only its sine and cosecant are positive, *i.e.* of the same sign as the corresponding co-functions of A.

Ex. To find the trigonometric functions of 120°.

$$120° = 90°+30°.$$

Therefore, by (37), sin, cos, tan, etc., of 120° are numerically = cos, sin, cot, etc., of 30°. Hence,

$$\sin 120° = \cos 30° = \frac{\sqrt{3}}{2}, \quad \cos 120° = -\sin 30° = -\frac{1}{2},$$
$$\tan 120° = -\cot 30° = -\sqrt{3}, \text{ etc.}$$

93. To compare the trigonometric functions of the angles A, $180° \pm A$, $360° \pm A$.

Let $\angle$ **AOP** $= A$. Produce **PO** to **R**, and draw **QOS** at an equal inclination to **OA** on the opposite side of it.

Then the angles described by a line revolving from **OA** (counter-clockwise) to **OP, OQ, OR, OS,** and **OP** again are

$$A, \quad 180°-A, \quad 180°+A, \quad 360°-A, \quad 360°+A.$$

Cut off equal radii

$$\mathbf{OP} = \mathbf{OQ} = \mathbf{OR} = \mathbf{OS}.$$

Then the fundamental triangles **OMP, ONQ, ONR, OMS** are equal in all respects; hence the trigonometric functions of the several angles are equal in magnitude and only differ in sign.

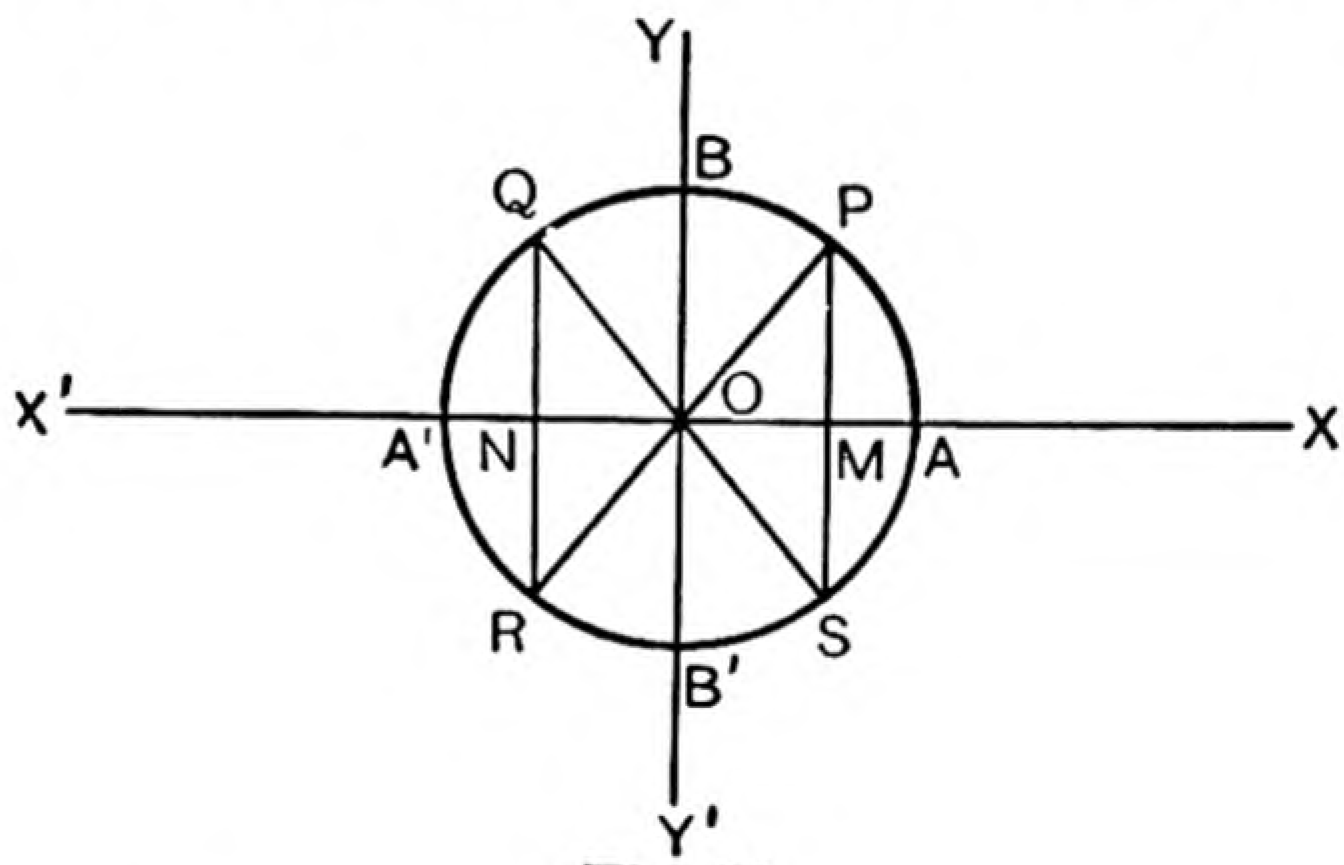


Fig. 81.

By means of the figure the signs may be readily decided, and we have the following results:—

$$\left.\begin{array}{l} \sin A = \sin(180°-A) = -\sin(180°+A) = -\sin(360°-A) \\ \qquad = (\sin 360°+A) \\ \cos A = -\cos(180°-A) = -\cos(180°+A) = +\cos(360°-A) \\ \qquad = (\cos 360°+A) \\ \tan A = -\tan(180°-A) = +\tan(180°+A) = -\tan(360°-A) \\ \qquad = \tan(360°+A) \end{array}\right\} \textbf{(38)}$$

The result thus established, viz. that *angles whose sum or difference is* $180°$ *or* $360°$ *have their trigonometric functions numerically equal*, will be easy to remember, the *signs* being settled by § 39, taking A in the first quadrant.

Corresponding results hold for the cosecant, secant, and cotangent, respectively.

ILLUSTRATIVE EXERCISES.

1. Write down the formulae connecting the cosecant, secant, and cotangent of A with those of the four angles $180° \pm A$, $360° \pm A$.

2. Express the formulae of the present article in circular measure, α being the circular measure of A.

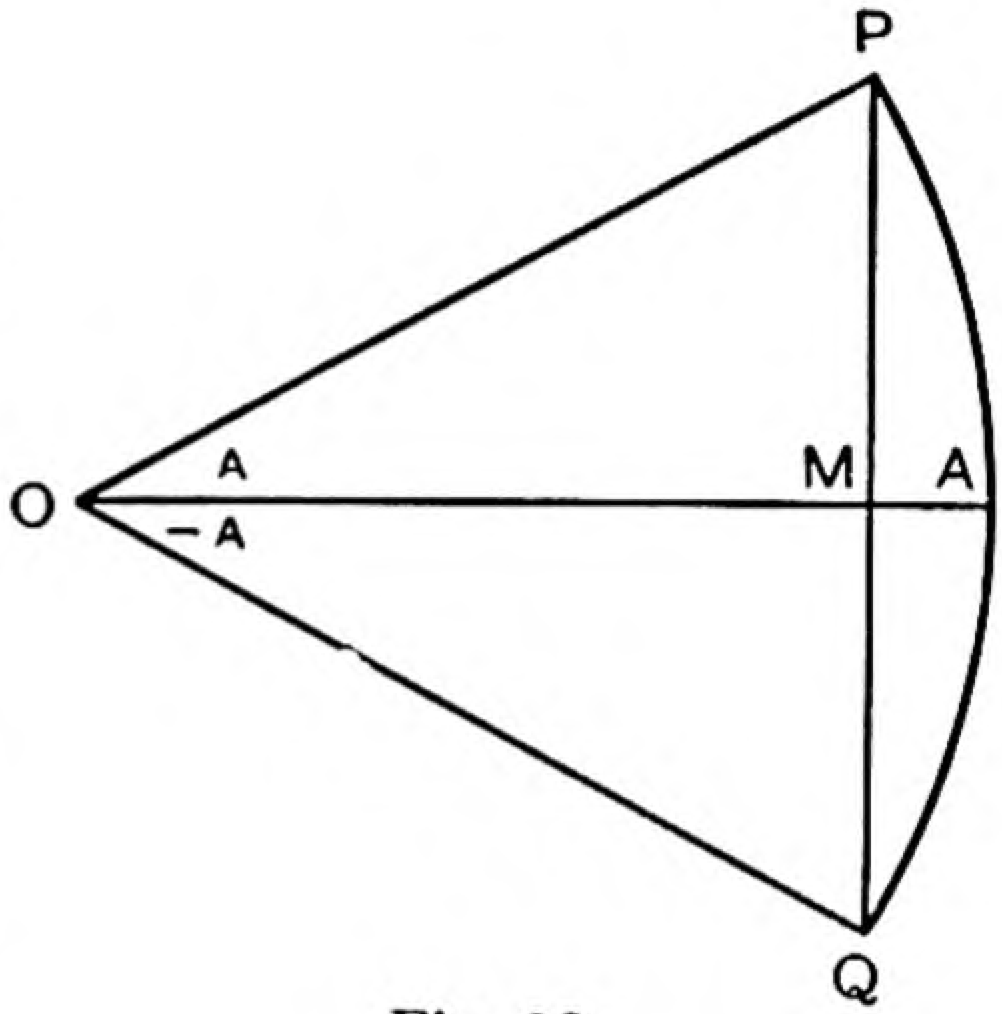


Fig. 82.

94. To compare the trigonometric functions of A and $-A$.

Let

$\angle \mathbf{AOP} = +A, \quad \angle \mathbf{AOQ} = -A.$

Then **OP, OQ** make equal angles with **OA** on opposite sides of **OA**.

Take **OP** = **OQ**; then the fundamental triangles **OMP, OMQ** will have a common base **OM**, also **MQ** = − **MP**.

$$\left.\begin{aligned} \therefore\quad \sin(-A) &= \frac{\mathrm{MQ}}{\mathrm{OQ}} = -\frac{\mathrm{MP}}{\mathrm{OP}} = -\sin A \\ \cos(-A) &= \frac{\mathrm{OM}}{\mathrm{OQ}} = \frac{\mathrm{OM}}{\mathrm{OP}} = \cos A \\ \tan(-A) &= \frac{\mathrm{MQ}}{\mathrm{OM}} = -\frac{\mathrm{MP}}{\mathrm{OM}} = -\tan A \\ \sec(-A) &= \frac{\mathrm{OQ}}{\mathrm{OM}} = \frac{\mathrm{OP}}{\mathrm{OM}} = \sec A \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (39)$$

and so on.

Thus the cosine and secant of $-A$ are equal to those of A, respectively, while the other functions of $-A$ are equal to those of $+A$, but of opposite sign.

95. To compare the trigonometric functions of A and $180° + A$.

[This has been done in § 93; but, as the case is an important one, we give a separate investigation.]

In Fig. 83, let

$$\angle \mathbf{AOP} = A, \quad \angle \mathbf{AOR} = 180°+A.$$

Take $\mathbf{OP} = \mathbf{OR}$, and complete the fundamental triangles **OMP, ONR**; then, *algebraically*,

$$\mathbf{ON} = -\mathbf{OM} \text{ and } \mathbf{NR} = -\mathbf{MP};$$

but

$$\mathbf{OR} = \mathbf{OP};$$

$$\therefore \quad \tan(180°+A) = \frac{\mathbf{NR}}{\mathbf{ON}} = \frac{-\mathbf{MP}}{-\mathbf{OM}} = \frac{\mathbf{MP}}{\mathbf{OM}} = \tan A \quad \ldots \text{(38A)}$$

Similarly, $\quad \cot(180°+A) = \cot A,$

while the other functions of $180°+A$ are equal and opposite in sign to the corresponding functions of A.

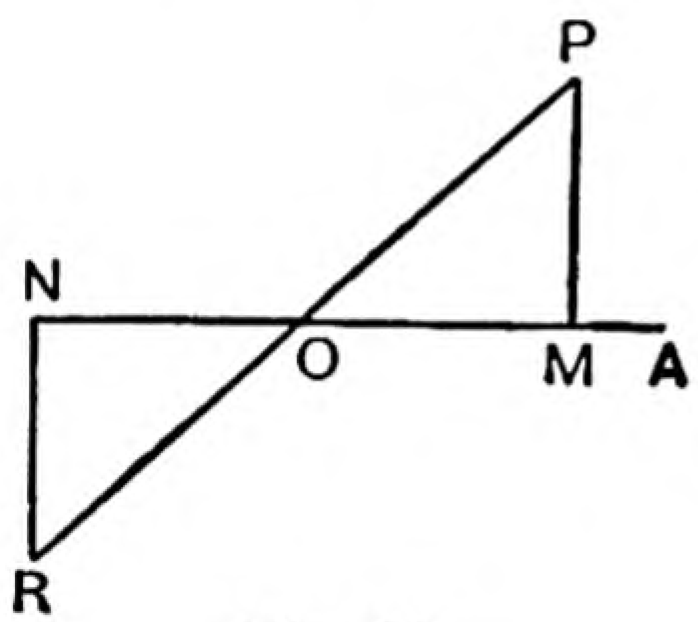


Fig. 83.

If $\angle A$ is obtuse, Fig. 36 (p. 42) will suggest the necessary figure, taking $\angle$**AOR** to be A.

We now add a few examples showing how to find the trigonometric functions of angles in any quadrant.

Ex. 1. To find cos 330° and cosec 330°.

Let a line describe 330° in revolving in the positive direction from **OA** to **OQ**. Then, since $330° = 360°-30°$, the radii **OP, OQ** bounding the angles 330° and 30° make equal angles on opposite sides with the primitive line, and, if $\mathbf{OP} = \mathbf{OQ}$, the fundamental triangles will have a common base **OM**, and **MQ** numerically $=$ **MP**. Hence the functions of 330° are numerically equal those of 30°.

Since 330° is in the fourth quadrant, its cosine and secant are alone positive;

$$\therefore \quad \cos 330° = \cos 30° = \frac{\sqrt{3}}{2},$$

$$\operatorname{cosec} 330° = -\operatorname{cosec} 30° = -2.$$

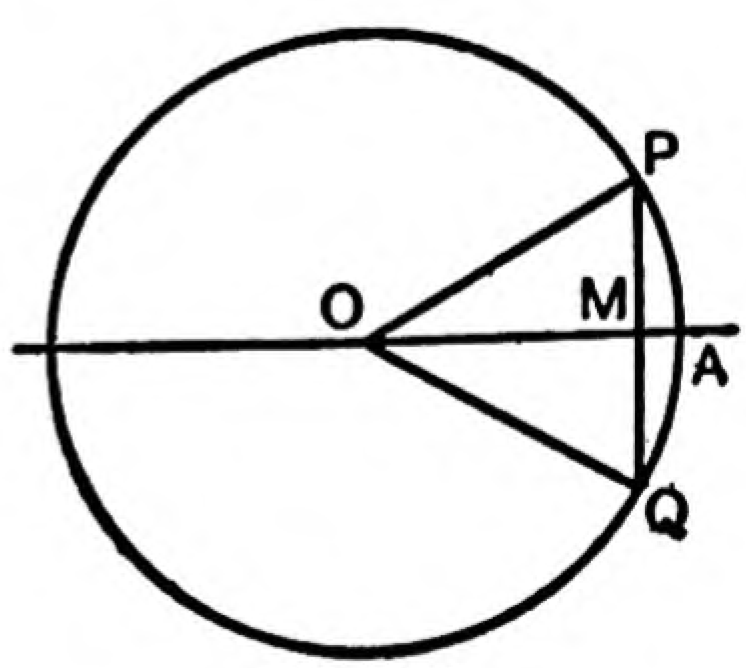


Fig. 84.

Another method would be to express the functions of 330° in terms of those of 60° by means of the identity $330° = 270°+60°$. We should now find

$$\cos 330° = \sin 60°$$

and

$$\operatorname{cosec} 330° = -\sec 60°.$$

Ex. 2. To find $\sin \frac{5}{4}\pi$ and $\cot \frac{5}{4}\pi$.

$$\tfrac{5}{4}\pi = \pi + \tfrac{1}{4}\pi;$$

hence the related angle is $\frac{1}{4}\pi$, and, if we make $\mathbf{OP} = \mathbf{OR}$, the fundamental

triangles are equal in all respects, as shown in Fig. 83A. Moreover, $\frac{5}{4}\pi$ is in the third quadrant, so that its tangent and cotangent are alone positive.

$$\therefore \quad \sin\frac{5\pi}{4} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

and $$\cot\frac{5\pi}{4} = \cot\frac{\pi}{4} = 1.$$

If we had employed the identity $\frac{5}{4}\pi = \frac{3}{2}\pi - \frac{1}{4}\pi$, we should have obtained the functions of $\frac{5}{4}\pi$ in terms of the *co-functions* of $\frac{1}{4}\pi$.

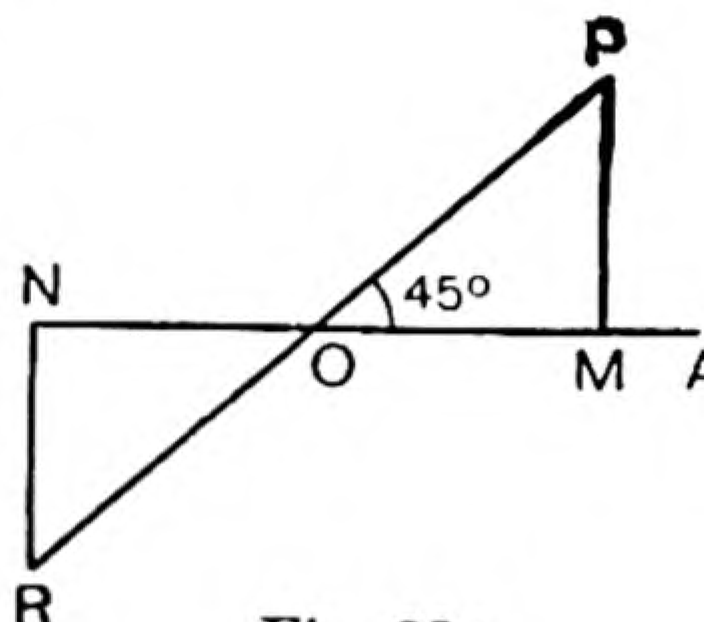


Fig. 83A.

Hitherto the angles with which we have been chiefly concerned have been intermediate between 0° and 360° in magnitude. If we require to find the ratios of any angle however great, positive or negative, lying beyond these limits, the following important proposition enables us to replace the given angle by a positive angle less than four right angles:—

96. The trigonometric functions of an angle are all unaltered when the angle is increased or decreased by four right angles or by any multiple of four right angles.

Let a line revolve through the $\angle \mathbf{XOP} = A$. Then, if it subsequently revolves about **O** in the positive or negative direction from **OP** through any number of complete revolutions, *i.e.* through any multiple of four right angles, it will again come into the same final position **OP**. Hence its fundamental triangle **OMP** is the same as that of the original angle A, and therefore the trigonometric functions are the same as those of A, both in magnitude and sign. Thus, *e.g.*—

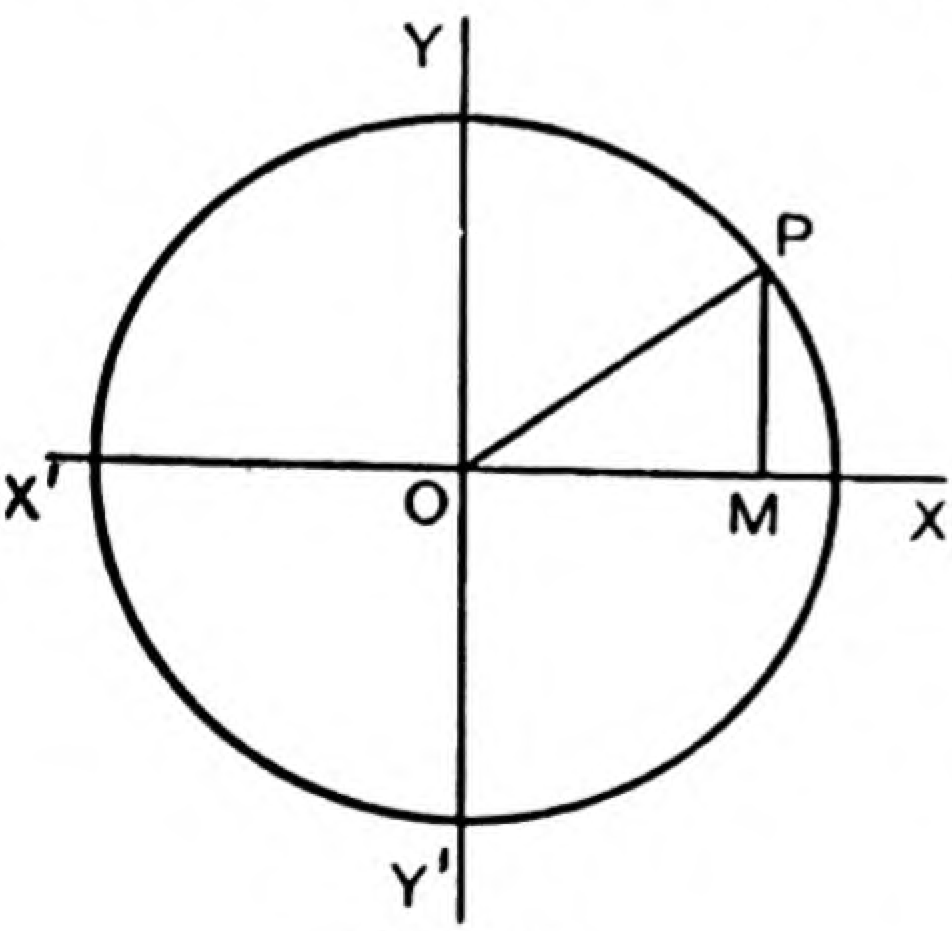


Fig. 85.

$$\left.\begin{matrix}\sin, \cos, \text{ or } \tan\,(360° + A) = \sin, \cos, \text{ or } \tan A, \\ \text{respectively}\end{matrix}\right\} \quad \textbf{(40)}$$

Def.—Angles described by revolving lines which start from the same initial position and arrive at the same final position are said to be **coterminal**.

Hence angles which differ by a multiple of four right angles are coterminal, and they have the same trigonometric functions.

The angles which are coterminal with A are

$$A-360°,\quad A+360°,\quad A+720°, \ldots \text{ and so on.}$$

and these are all included in the general expression

$$A \pm n.360°,$$

where n is any integer.

In circular measure all angles of the form $\alpha \pm 2n\pi$ are coterminal with α.

97. To find the functions of any angle, however great, positive or negative, we now have the following rules:—

1st. *Add or subtract a multiple of* 360° (*or* 2π), *such as to make the result positive and* $< 360°$.

All the trigonometric functions will be unaltered.

2nd. *Find a related angle, in the first quadrant,* i.e. *an angle which, added to or subtracted from the given angle, gives a multiple of* 180° (*or* π). [This may be done by finding the multiple of 180° which is nearest to the given angle, greater or less than it, and subtracting one from the other.]

3rd. *The trigonometric functions are the same numerically for the original and related angles,* and they therefore only differ in sign (§ 93).

4th. *The signs of the functions are determined by* § 39.

Note.—The required trigonometric functions are thus made to depend on those of a related angle in the first quadrant. *If the original angle be any multiple of* 30° or 45°, the related angle will be either 30°, 45°, or 60°, and hence the functions can be written down from Chapter VI. In other cases it would be usually necessary to refer to a book of tables. The table on page 21 may be referred to if the related angle be any multiple of 5°.

Ex. 1. Thus $\cos 405° = \cos(360°+45°) = \cos 45°$

$$= \frac{1}{\sqrt{2}}.$$

Ex. 2. To find cosec 1830°.

Divide 1830° by 360°; quotient is 5, and remainder 30°.

$$\therefore \quad \operatorname{cosec} 1830° = \operatorname{cosec} (5 \times 360° + 30°) = \operatorname{cosec} 30° = 2.$$

Ex. 3. To find $\cot(-\frac{5}{3}\pi)$.

Since $\frac{5}{3}\pi < 2\pi$, we add 2π, or four right angles, thus—

$$\cot(-\tfrac{5}{3}\pi) = \cot(2\pi - \tfrac{5}{3}\pi) = \cot \tfrac{1}{3}\pi$$
$$= \tfrac{1}{3}\sqrt{3} \text{ (since } \tfrac{1}{3}\pi \text{ radians} = 60°).$$

Ex. 4. $$\sin(-750°) = \sin(-720° - 30°) = (-30°)$$
$$= \sin(360° - 30°).$$

The new angle is in the fourth quadrant; hence its sine is negative and $= -\sin 30° = -\frac{1}{2}$.

Ex. 5. $$\tan 690° = \tan(360° + 330°) = \tan 330° = \tan(360° - 30°)$$
$$= -\tan 30° = -\frac{1}{\sqrt{3}}.$$

We saw in § 54 that the cosine curve was simply the sine curve shifted through a distance $\frac{1}{2}\pi$ or 90° towards the left. From this follows the relation $\cos\theta = \sin(\theta + 90°)$. If the cosine curve is now shifted a distance $\frac{1}{2}\pi$ to the left, we obtain a curve which is the sine curve upside down, *i.e.* corresponding ordinates of it and the sine curve are equal in magnitude but opposite in sign. From this follows the relation $-\sin\theta = \cos(\theta + 90°)$.

Since altogether the sine curve has been shifted a distance π or 180° to the left, the last relation also gives $-\sin\theta = \sin(\theta + 180°)$. Similarly the relations between the other trigonometrical functions of θ and $\theta + 180°$ can be verified from the graphs of Chap. V.

98. The following examples are instructive:—

Ex. 1. To draw the curve whose equation is

$$y/a = \sec x/a + \operatorname{cosec} x/a.$$

When $x = 0$, $\quad y/a = 1 + \infty = \infty.$

When x lies between 0 and $\frac{1}{2}\pi a$, both $\sec x/a$ and $\operatorname{cosec} x/a$ are finite and positive; hence y is finite and positive, and the curve lies above the horizontal axis.

When $x = \frac{1}{4}\pi a$, $\quad y/a = \sqrt{2} + \sqrt{2}$ or $y = 2\sqrt{2}a$.

When x passes through the value $\frac{1}{2}\pi a$, $\sec x/a$ becomes infinite and suddenly changes from $+\infty$ to $-\infty$; hence y/a changes from $1+\infty$ to $1-\infty$, that is, from $+\infty$ to $-\infty$ (since the addition of 1 to $-\infty$ has no practical effect on the result, which is still negative and infinite).

When $x = \frac{3}{4}\pi a$, $\qquad y/a = -\sqrt{2}+\sqrt{2} = 0,$

and the curve cuts the horizontal axis.

When $x = \pi a$, $\qquad \sec x/a = -1,$

and $\operatorname{cosec} x/a$ changes from $+\infty$ to $-\infty$; hence y changes from $+\infty$ to $-\infty$.

When $x = \frac{5}{4}\pi a$, $\qquad y = -2\sqrt{2}a,$

and, when x passes through the value $\frac{3}{2}\pi a$, $\sec x/a$ becomes infinite, and suddenly changes from $-\infty$ to $+\infty$; so that y changes from $-\infty$ to $+\infty$.

When $x = \frac{7}{4}\pi a$,

$$y/a = \sqrt{2}-\sqrt{2} = 0,$$

and the curve cuts the horizontal axis.

When $x = 2\pi a$,

$$\sec x/a = 1;$$

and $\operatorname{cosec} x/a$ changes from $-\infty$ to $+\infty$, and therefore y changes from $-\infty$ to $+\infty$.

Since both $\sec x/a$ and $\operatorname{cosec} x/a$ are unaltered when x is increased by any multiple of $2\pi a$, the values of y exhibited in the figure are repeated in the compartments $x = 2\pi a$ to $x = 4\pi a$, $x = 4\pi a$ to $x = 6\pi a$, and so on.

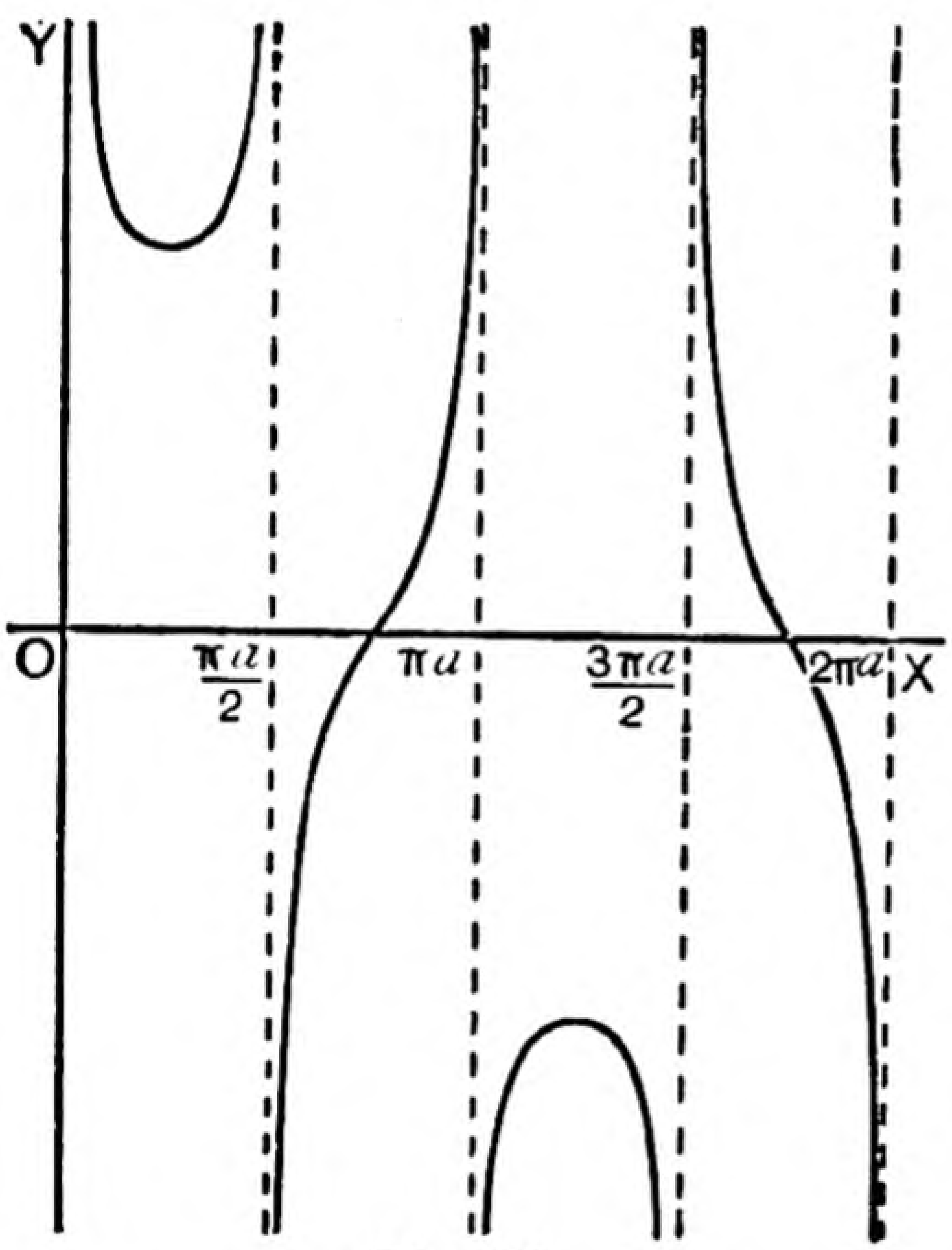


Fig. 86.

Ex. 2. To show that $\sin\theta$ is greater than $2\theta/\pi$, if θ lie between 0 and $\frac{1}{2}\pi$.

Draw the curve of sines, as in Art. 53.

Then, if $ON = \frac{1}{2}\pi$, $QN = 1$; and, if $OM = \theta$, $LM = \sin\theta$. (Fig. 87)

Join OQ, meeting LM in P. Then, by similar triangles,

$$\frac{PM}{OM} = \frac{QN}{ON},$$

and therefore $\qquad PM = \frac{2\theta}{\pi};$

and, since $LM > PM$, $\qquad \therefore\ \sin\theta > \frac{2\theta}{\pi}.$

If $\theta = 0$, or $\frac{1}{2}\pi$, the quantities are equal, by §§ 64, 65.
Between the given limits, $\sin \theta$ lies between unity and $2\theta/\pi$.
If we write $\frac{1}{2}\pi - \theta$ for θ, then, between the same limits, we see that

$$\cos \theta \text{ lies between unity and } 1 - \frac{2\theta}{\pi},$$

and hence
$$\tan \theta < \frac{\pi}{\pi - 2\theta},$$
θ lying between 0 and $\frac{1}{2}\pi$.

Caution.—We would again call attention to the importance of taking account of the changes of sign when a function becomes infinite, especially in tracing curves such as that of Ex. 1 above. Unless extreme care is exercised, there is great danger of drawing the infinite branches on the wrong side of the horizontal axis.

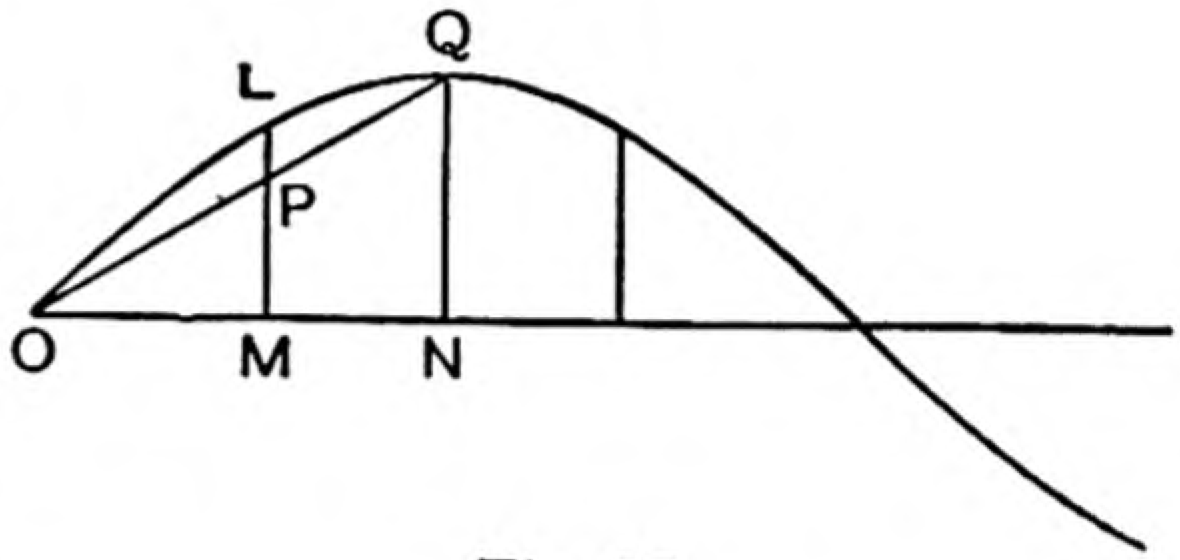


Fig. 87.

Note also that $1 - \infty$ is $-\infty$, not $+\infty$; and $\infty - 1$ is $+\infty$, not $-\infty$. But it must not be inferred that, because $1 + \infty = \infty$, therefore $1 = 0$, for two quantities may both be infinite, and yet differ by unity.

EXAMPLES VIII.

1. Show that, whatever be the magnitude of the angle A,
$$\sin A = \cos (90° - A).$$

2. Prove that $\qquad \tan (180° - A) = -\tan A.$

3. Show, by means of a diagram, that, if A lie between three and four right angles, then $\cos (180° - A) = -\cos A$.

4. Prove from a figure that
$$\sec (270° - A) = -\operatorname{cosec} A, \qquad \cot (360° - A) = -\cot A,$$
$$\operatorname{cosec} (270° + A) = -\sec A, \qquad \sec (540° + A) = -\sec A.$$

5. Prove that in general
$$\cos A = \cos (2n\,180° + A) = -\cos (\overline{2n+1}\,180° - A)$$
$$= -\cos (\overline{2n+1}\,180° + A) = \cos (2n\,180° - A).$$

6. Prove that $\operatorname{vers} (270° + A) \operatorname{vers} (270° - A) = \cos^2 A$.

7. Given that $\tan 21° 48' = \cdot 400$, find to 3 places of decimals the numerical value of $\tan 68° 12'$, $\cos 21° 48'$, and $\sin 21° 48'$.

8. Find $\sin 300°$, $\tan 855°$, and $\cot 435°$.

9. Prove that, if the sum or difference of two angles is an odd number of right angles, the functions of one angle are numerically equal to the

co-functions of the other. Under what conditions are the equalities *all* algebraic equalities ?

10. Find sec $210°$, tan $(-480°)$, sin $990°$.

11. Find cos $840°$, cosec $945°$, cot $(-3660°)$.

12. Find cos $1035°$, tan $1830°$, vers $6420°$.

13. Find $\sin\frac{5\pi}{3}$, $\operatorname{cosec}\frac{15\pi}{4}$, $\tan\frac{11\pi}{3}$.

14. Find $\operatorname{vers}\frac{13\pi}{4}$, $\cot\frac{37\pi}{6}$, $\operatorname{cosec}\frac{19\pi}{3}$.

15. Find $\sec\left(\frac{-3\pi}{4}\right)$, $\operatorname{covers}\left(\frac{-14\pi}{3}\right)$, $\cos\left(\frac{-5\pi}{2}\right)$.

*16. Find general expressions for the limits between which must lie all angles whose cosines are algebraically less than their sines.

17. Given $\sin 36° 53' = \cdot 6$ and $\cos 36° 53' = \cdot 8$, find the angle whose sine $= -\cdot 6$ and cosine is $\cdot 8$, and also the angle whose sine is $\cdot 6$ and cosine $-\cdot 8$.

18. Trace the changes in the value of $\frac{\cos 2A}{\cos A}$ as A goes from 0 to 180°.

19. Trace the changes in the value of $\sin\theta + \sqrt{3}\cos\theta$ as θ changes from 0° to 180°.

CHAPTER IX.

INVERSE FUNCTIONS.

99. It often happens that, instead of an angle being given, its sine, cosine, or some other function is given, and it is therefore convenient to have a notation for representing an angle, one of whose trigonometric functions has a given value.

DEF.—The angle whose sine is a given number m is called the **inverse sine of that number,** and is written $\sin^{-1} m$. In like manner, the angle whose cosine is a given number n is called the **inverse cosine of that number** and written $\cos^{-1} n$, and similarly for the other functions.

Thus the statement that $\sin 30° = \frac{1}{2}$ is also written in the form $\sin^{-1} \frac{1}{2} = 30°$, and this is read "inverse sine $\frac{1}{2}$ equals 30°," or "*the angle whose sine is* $\frac{1}{2}$ equals 30°."

Similarly, since $\tan 45° = 1$, $\therefore \tan^{-1} 1 = 45°$;
and, since $\sec 60° = 2$; $\therefore \sec^{-1} 2 = 60°$.

The inverse notation leads at once to the identities

$$\left.\begin{array}{c} \sin(\sin^{-1} m) = m, \qquad \cos(\cos^{-1} n) = n, \\ \tan(\tan^{-1} k) = k, \text{ etc.} \\ \theta = \sin^{-1}(\sin\theta) = \cos^{-1}(\cos\theta) = \tan^{-1}(\tan\theta) = \text{etc.} \end{array}\right\} \ldots\textbf{(41)}$$

CAUTION 1.—The -1 in such expressions as $\sin^{-1}x$ is *not of the nature of an index*, thus differing essentially from the 2 in $\sin^2\theta$. This is, of course, illogical, and it would be easy for anyone to devise a better notation (*e.g.* to write the -1 below the line thus, $\sin_{-1}x$). But the above notation is almost universally used in this country. Continental writers use the word *arc* to denote inverse functions, thus, arc $\tan x$, and so on, the *angle* which has any given function being, of course, proportional to the arc which it subtends at the centre.

CAUTION 2.—In such expressions as $\sin^{-1} x$, it must be borne in mind that x is not an angle, but a number (positive or negative). $\text{Sin}^{-1}x$ is itself an angle. It would be an easy slip to speak of the inverse sine of an angle, but this, of course, is absurd.

100. Expression of trigonometric identities in inverse notation.—If we are given any identical relation connecting any two trigonometric functions of the same or different angles, we may always express it in terms of the corresponding inverse functions. This may best be illustrated by a few examples.

Ex. 1. To prove the identities

$$\text{(i)}\ \sec^{-1} x = \cos^{-1}\frac{1}{x}; \qquad \text{(ii)}\ \cos^{-1} x = \sin^{-1}\sqrt{(1-x^2)};$$

$$\text{(iii)}\ \cot^{-1} x = \operatorname{cosec}^{-1}\sqrt{(1+x^2)}.$$

In the identity $\qquad \sec\theta = \dfrac{1}{\cos\theta},$

put $\sec\theta = x$, so that $\qquad \theta = \sec^{-1} x.$

Then $\qquad \cos\theta = \dfrac{1}{\sec\theta} = \dfrac{1}{x}; \qquad \therefore\ \cos^{-1}\dfrac{1}{x} = \theta = \sec^{-1} x.$

In the identity $\qquad \cos^2\theta + \sin^2\theta = 1,$

put $\cos\theta = x$, so that $\qquad \theta = \cos^{-1} x.$

Then $\qquad \sin\theta = \sqrt{(1-\cos^2\theta)} = \sqrt{(1-x^2)};$

$$\therefore\ \sin^{-1}\sqrt{(1-x^2)} = \theta = \cos^{-1} x.$$

In the identity $\qquad \operatorname{cosec}^2\theta = 1+\cot^2\theta,$

put $\cot\theta = x$, so that $\qquad \theta = \cot^{-1} x.$

Then $\qquad \operatorname{cosec}\theta = \sqrt{(1+\cot^2\theta)} = \sqrt{(1+x^2)};$

$$\therefore\ \operatorname{cosec}^{-1}\sqrt{(1+x^2)} = \theta = \cot^{-1} x.$$

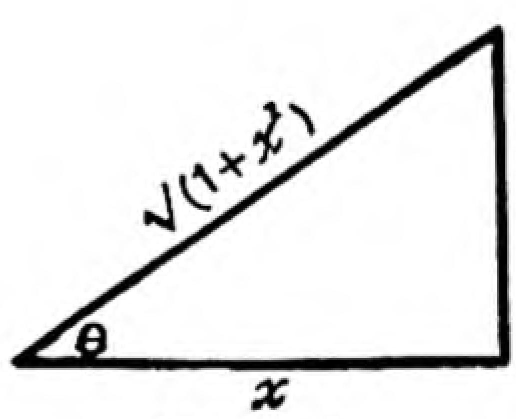


Fig. 88.

Ex. 2. To express $\cot^{-1} x$ as an inverse cosine.

Let $\qquad \theta = \cot^{-1} x; \qquad \therefore\ x = \cot\theta.$

Drawing the fundamental triangle as in § 78, we have

$$\cos\theta = \frac{x}{\sqrt{(1+x^2)}};$$

$$\therefore\ \theta \text{ or } \cot^{-1} x = \cos^{-1}\frac{x}{\sqrt{(1+x^2)}}.$$

Ex. 3. To prove that $\tan^{-1} x + \cot^{-1} x = 90°$, x being positive, and $\tan^{-1} x$, $\cot^{-1} x$ being acute angles.

In the identity $\tan(90° - A) = \cot A$,

put $\cot A = x$, so that $\tan(90° - A) = x$;

$$\therefore \quad A = \cot^{-1} x \text{ and } 90° - A = \tan^{-1} x;$$

$$\therefore \quad \tan^{-1} x + \cot^{-1} x = 90°.$$

101. The student will find it an instructive exercise to translate the results given in the table on p. 80 into inverse notation. The results are given on the next page, but it should be noted that the *rows* of this table are written down from the *columns* of the previous one, and *vice versa*.

102. **Principal values of inverse functions.**

The constructions in §§ 42-44 give in every case *two* angles with different boundaries, in each of which one of the trigonometric functions has a given value; moreover, any angle coterminal with one of these two will also have the same trigonometric functions, and will also be a solution of the problem. It hence follows that an angle is not completely specified when one of its functions is given, and therefore that such expressions as $\sin^{-1} m$, $\tan^{-1} k$, $\sec^{-1} l$, etc., each admit of any number of different possible values. It is convenient in every case to distinguish one of these values as the *principal value* of the inverse function. Before defining this we shall give a few simple illustrations.

Suppose, in the first place, that one of the functions of an angle has a given positive value, say $\tan\theta = 1$. Since $\tan(180° + A) = \tan A$ and $\tan 45° = 1$, we know that A may be either $45°$ or $180° + 45°$, that is, $225°$ or any angle formed by adding or subtracting a multiple of $180°$, and so on. But it is usual to regard the *smallest angle*, viz. $45°$, as the *principal value* of $\tan^{-1} 1$. In general, when the given function is positive, the principal value of the inverse function is taken to be the least positive angle satisfying the given conditions, and this always an angle in the first quadrant.

Next, suppose the given function is negative.

If, *e.g.* $\tan\theta = -1$, θ may be either $= -45°$, or $= 135°$, or $= 315°$. In this case it is usual to regard the *numerically least angle*, $-45°$, as the *principal value* of $\tan^{-1}(-1)$.

Similarly the *principal value* of $\sin^{-1}(-\frac{1}{2})$ is $-30°$, this being its numerically least value.

If $\cos\theta = (-\frac{1}{2})$, the least positive and negative values of θ are $120°$ and $-120°$, and are *equal*. In such cases we take the *positive angle*, $120°$, as the *principal value* of $\cos^{-1}(-\frac{1}{2})$.

$$
\begin{aligned}
\sin^{-1}x &= \sin^{-1} x = \cos^{-1}\sqrt{(1-x^2)} = \tan^{-1}\frac{x}{\sqrt{(1-x^2)}} = \cot^{-1}\frac{\sqrt{(1-x^2)}}{x} = \sec^{-1}\frac{1}{\sqrt{(1-x^2)}} = \operatorname{cosec}^{-1}\frac{1}{x};\\
\cos^{-1}x &= \sin^{-1}\sqrt{(1-x^2)} = \cos^{-1} x = \tan^{-1}\frac{\sqrt{(1-x^2)}}{x} = \cot^{-1}\frac{x}{\sqrt{(1-x^2)}} = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\frac{1}{\sqrt{(1-x^2)}};\\
\tan^{-1}x &= \sin^{-1}\frac{x}{\sqrt{(1+x^2)}} = \cos^{-1}\frac{1}{\sqrt{(1+x^2)}} = \tan^{-1} x = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{(1+x^2)} = \operatorname{cosec}^{-1}\frac{\sqrt{(1+x^2)}}{x};\\
\cot^{-1}x &= \sin^{-1}\frac{1}{\sqrt{(1+x^2)}} = \cos^{-1}\frac{x}{\sqrt{(1+x^2)}} = \tan^{-1}\frac{1}{x} = \cot^{-1} x = \sec^{-1}\frac{\sqrt{(1+x^2)}}{x} = \operatorname{cosec}^{-1}\sqrt{(1+x^2)};\\
\sec^{-1}x &= \sin^{-1}\frac{\sqrt{(x^2-1)}}{x} \pm \cos^{-1}\frac{1}{x} = \tan^{-1}\sqrt{(x^2-1)} = \cot^{-1}\frac{1}{\sqrt{(x^2-1)}} = \sec^{-1} x = \operatorname{cosec}^{-1}\frac{x}{\sqrt{(x^2-1)}};\\
\operatorname{cosec}^{-1}x &= \sin^{-1}\frac{1}{x} = \cos^{-1}\frac{\sqrt{(x^2-1)}}{x} = \tan^{-1}\frac{1}{\sqrt{(x^2-1)}} = \cot^{-1}\sqrt{(x^2-1)} = \sec^{-1}\frac{x}{\sqrt{(x^2-1)}} = \operatorname{cosec}^{-1} x.
\end{aligned}
$$

We may now give the following definition:—

DEF.—The **principal value** of an inverse function of a given number is the numerically least angle having that number for its function, the positive angle being chosen in cases of equality.

103. Meaning of $(-1)^n$.—In the following article we shall introduce the symbol $(-1)^n$ in a form with which the student may perhaps be unfamiliar. If we form the successive powers of -1, beginning with the first, we have

$$(-1)^1=-1,\ (-1)^2=+1,\ (-1)^3=-1,\ (-1)^4=+1,$$

and so on. Similarly, by the theory of zero and negative indices in Algebra,

$$(-1)^0=1,\ (-1)^{-1}=\frac{1}{-1}=-1,\ (-1)^{-2}=\frac{1}{(-1)^2}=1,$$

$$(-1)^{-3}=\frac{1}{(-1)^3}=-1,$$

and so on. Thus every *even* positive or negative power of -1 is equal to $+1$, and every *odd* power is equal to -1; that is, $(-1)^n$ equals positive or negative unity according as n is even or odd.

Hence the factor $(-1)^n$ affords a convenient means of indicating the sign of the nth of a series of alternately negative and positive quantities.

104. To find a general expression for all angles which have a given sine or cosecant.

(i) Let the given *sine* $=x$. Let A be the principal value of $\sin^{-1}x$, *i.e.* the smallest positive or negative angle whose sine is x. Then, since

$$\sin(180°-A)=\sin A,$$

$\therefore$ $180°-A$ is another angle whose sine is x.

Let these two angles be **AOP**, **AOQ** in Fig. 89.

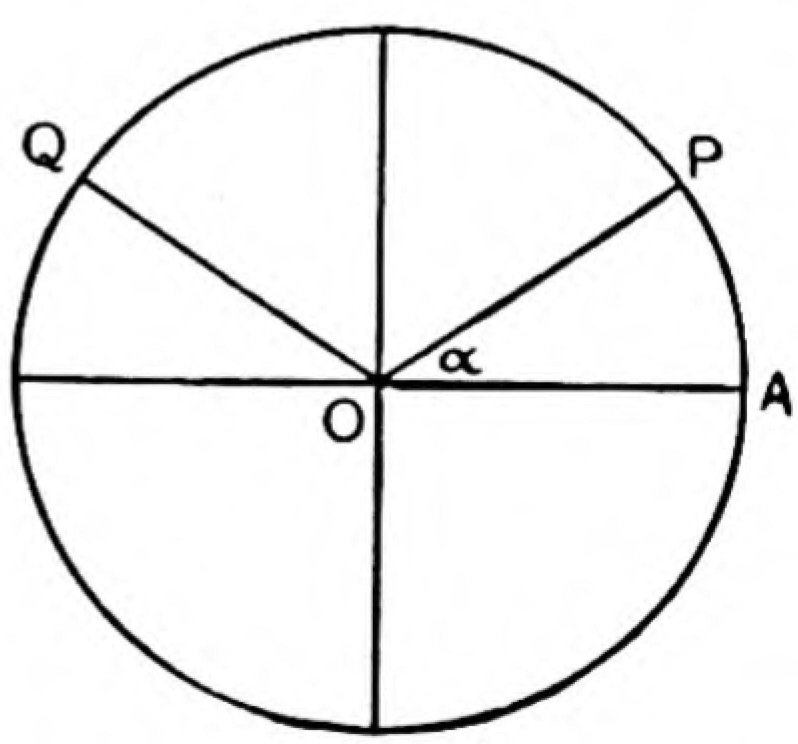


Fig. 89.

Then any angle coterminal with either of these will also have the same sine, viz. x. And since coterminal angles differ by a multiple of 360° (§ 96), the required angles will be obtained by adding positive or negative multiples of 360° to A or $180°-A$. They will therefore belong to either of the two series obtained by

assigning to m different positive or negative integral values (including zero) in

$$m.360°+A \quad \text{or} \quad m.360°+180°-A;$$

or, as we may write them,

$$2m.180°+A \quad \text{or} \quad (2m+1)\,180°-A,$$

where m is any positive or negative integer.

Now $2m.180°$ and $(2m+1)\,180°$ are both multiples of 180°, but we notice that in the first form the multiple is *even* and A is *added*, and in the second the multiple is *odd* and A is subtracted. Thus A is taken with a positive or negative sign according as it is associated with an even or odd multiple of 180°. Hence, from the last article, all the angles which have the same sine as A are obtained by giving different positive or negative integral values (including zero) to n in the formula

$$\mathbf{n.180°+(-1)^n A} \text{ .... } \mathbf{(42)}$$

It is more usual to express the result in circular measure, and the general expression for all angles which have the same sine as α is then seen to be

$$\mathbf{n\pi+(-1)^n\,\alpha} \text{ .... } \mathbf{(42\pi)}$$

[Before proceeding further, these results should be verified by giving different values to n; thus, if

$n=$	1,	2,	3,	4...,
the angle $=$	$180°-A$,	$360°+A$,	$540°-A$,	$720°+A$....
If $n=$	0,	-1,	-2,	-3...,
the angle $=$	A,	$-180°-A$,	$-360°+A$,	$-540°-A$....]

(ii) Since the *cosecant* is the reciprocal of the sine, if $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$, then $\sin\theta = \sin\alpha$; *i.e.* angles which have the same cosecant have the same sine, and *vice versa.* Hence the general expressions

$$n.180°+(-1)^n A \text{ (degrees)................. } (42)$$

$$n\pi+(-1)^n\,\alpha \text{ (radians)................. } (42\pi)$$

also represent all the angles having the same cosecants as A and α respectively.

Note.—Since covers $\theta = 1-\sin\theta$, the same expressions also represent all the angles which have a given *coversed sine.*

Ex. 1. To find the general expression for all angles whose sine is $\frac{1}{2}$.

The principal value of $\sin^{-1}\frac{1}{2}$ is $30°$. Hence, the general expression is $n.180°+(-1)^n\,30°$ or, in radians, $n\pi+(-1)^n.\pi/6$.

CAUTION.—Such hybrid expressions as $n\pi+(-1)^n\,30°$ are incorrect.

Ex. 2. Verify that

$$n.360°+90°\pm(90°-A)$$

is another general formula representing all angles which have the same sine and cosecant as A where n is a positive or negative integer or zero.

If we take the lower sign in the ambiguity, we obtain $n.360°+A$, which represents all the angles coterminal with A, and therefore having the same sine as A.

If we take the upper sign, we obtain $n.360°+180°-A$, which represents all the angles coterminal with $180°-A$, and whose sine is therefore $=\sin(180°-A)=\sin A$, as required to be proved.

ILLUSTRATIVE EXERCISES.

1. Investigate, *in circular measure, ab initio*, the general expression for the circular measure of all angles which have the same sine as a given angle α. [Write out the investigation of § 104 (i), substituting π for $180°$, and using circular measure *throughout*.]

2. Also verify that $(2n+\frac{1}{2})\,\pi\pm(\frac{1}{2}\pi-\alpha)$ is a general expression satisfying these conditions.

3. Obtain the general expression for all angles whose sine is $-\frac{1}{2}\sqrt{3}$ in the form $n\pi+(-1)^{n+1}.\frac{1}{3}\pi$.

105. To find a general expression for all angles which have a given cosine or secant.

(i) Let the given *cosine* $=x$. Let A be the principal value of $\cos^{-1}x$, *i.e.* the smallest positive angle having its cosine $=x$.

Then, since $\qquad \cos(-A)=\cos A,$

$\therefore\quad -A$ is another angle whose cosine is x.

Also any angle coterminal with either of these will have the same cosine, viz. x. And since coterminal angles differ by a multiple of $360°$, we conclude that all the angles having the same cosine as A are obtained by giving different positive or negative integral values (including zero) to n, and taking either sign in the formula

$$\boldsymbol{n.360°\pm A}\text{........................} \quad \textbf{(43)}$$

In circular measure the general expression for all angles which have the same cosine as α is

$$2n\pi \pm \alpha \quad \text{.........................} \quad (43\pi)$$

This is the form usually remembered.

(ii) Since the *secant* is the reciprocal of the cosine, all angles which have the same secant also have the same cosine, and are therefore in the same general forms, viz.

$$n \,.\, 360^\circ \pm A \quad \text{(degrees)},$$

or

$$2n\pi \pm \alpha \quad \text{(radians)}.$$

NOTE.—Since vers $\theta = 1 - \cos\theta$, the same expressions also represent all angles which have the same versed sine.

Ex. To find the general expression for all angles whose secant $= -2$. If $\sec\theta = -2$, $\cos\theta = -\frac{1}{2} = -\cos 60^\circ = \cos(180^\circ - 60^\circ) = \cos 120^\circ$. Hence the general expression for θ is

$$\theta = n \,.\, 360^\circ \pm 120^\circ, \text{ or } \theta = 2n\pi \pm \tfrac{2}{3}\pi \text{ (radians)}.$$

ILLUSTRATIVE EXERCISE.

Investigate *ab initio*, in circular measure, the general expression for all angles having the same *secant* as a given angle α.

106. To find a general expression for all angles which have a given tangent or cotangent.

(i) Let the given *tangent* $= x$. Let A be the principal value of $\tan^{-1}x$, *i.e.* the smallest positive angle whose tangent is x. Then, since (§ 95)

$$\tan(180^\circ + A) = \tan A;$$

$$\therefore \quad 180^\circ + A \text{ is another angle whose tangent is } x.$$

Any angle coterminal with either of these will have the same tangent, viz. x. Now, if a radius vector, after describing the angle A, revolves through an even multiple of two right angles in either the positive or negative direction, it will have described an angle coterminal with A; but, if it revolves through an odd multiple of two right angles, the angle will be coterminal with $180^\circ + A$. Hence all angles having the same tangent as A differ from A by some positive or negative, odd or even multiple of 180°, and we conclude that they are

obtained by giving different positive or negative integral values (including zero) to n in the formula

$$n\,.\,180^\circ + A \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(44)}$$

In circular measure, the general expression for all angles which have the same tangent as a is

$$n\pi + a \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(44}\pi\textbf{)}$$

and this is the form usually remembered.

(ii) Since the *cotangent* is the reciprocal of the tangent, all angles having the same cotangent also have the same tangent, and are therefore included in the same general forms, viz.

$$x = n\,.\,180^\circ + A \quad \text{(degrees).}$$

or

$$n\pi + a \quad \text{(radians).}$$

Ex. The general form of all angles whose tangent is -1 is $n\pi - \frac{1}{4}\pi$, and the general form of all angles whose cotangent is 1 is $n\pi + \frac{1}{4}\pi$ (radians).

107. Simple Trigonometric Equations.—Summary.

Just as $x^2 = a^2$ is an algebraic equation in the unknown quantity x, having two roots given by the formula $x = \pm a$, so equations like $\sin\theta = \sin a$ may be regarded as **trigonometric equations** having an infinite number of roots, and the last three articles show us that the general formula in circular measure for the roots of the trigonometric equation

$$\left.\begin{array}{r}\sin\theta = \sin a\\ \text{or } \operatorname{cosec}\theta = \operatorname{cosec} a\end{array}\right\} \left\{\begin{array}{l}\text{is}\quad \theta = n\pi + (-1)^n a \;\ldots\ldots\ldots\ldots \textbf{(42)}\\ \text{or}\quad \theta = (2n+\frac{1}{2})\pi \pm (\frac{1}{2}\pi - a)\end{array}\right.$$

[§ 104. Ex. 2]

$$\left.\begin{array}{r}\cos\theta = \cos a\\ \text{or}\quad \sec\theta = \sec a\end{array}\right\} \quad \text{is}\quad \theta = 2n\pi \pm a \;\ldots\ldots\ldots\ldots\ldots \textbf{(43)}$$

$$\left.\begin{array}{r}\tan\theta = \tan a\\ \text{or}\quad \cot\theta = \cot a\end{array}\right\} \quad \text{is}\quad \theta = n\pi + a \;\ldots\ldots\ldots\ldots\ldots \textbf{(44)}$$

108. **If two of these hold,*** as, for instance, $\sin\theta = \sin a$,

* This often occurs in the process of solving trigonometric equations, as we shall find in the next chapter. The two equations must be *independent*, *i.e.* the functions they involve must not be reciprocals of each other; thus, the equations $\tan\theta = \tan a$ and $\cot\theta = \cot a$ are not independent.

and $\cos\theta = \cos\alpha$, the angles θ and α must be evidently coterminal, so that the only solution is

$$\theta = 2n\pi + \alpha. \qquad (45)$$

On the other hand, the equations

$$\sin^2\theta = \sin^2\alpha, \quad \cos^2\theta = \cos^2\alpha, \quad \text{or} \quad \tan^2\theta = \tan^2\alpha$$

become

$$\sin\theta = \pm\sin\alpha, \quad \cos\theta = \pm\cos\alpha, \quad \tan\theta = \pm\tan\alpha,$$

each of which is satisfied by the values

$$\theta = n\pi \pm \alpha. \qquad (46)$$

NOTE.—It will be found best, as a general rule, to regard the symbols of the inverse notation as denoting *principal values only*. Thus the general solution, *e.g.* of the equation $\sin\theta = \frac{3}{4}$ will be written $n\pi + (-1)^n \sin^{-1}\frac{3}{4}$. It would be quite as logical to assume that $\sin^{-1}\frac{3}{4}$ represented the general expression for *every angle* whose sine was $\frac{3}{4}$. Similarly, we might regard $\tan^{-1} 1$ as representing either $\frac{1}{4}\pi$ or the general expression $n\pi + \frac{1}{4}\pi$; but the first is the better plan on the whole. No definite rule, however, exists, and ambiguity *might* be avoided by writing general solutions thus, *e.g.*

$$n\pi + (-1)^n \text{ (princ. val. } \sin^{-1}\tfrac{3}{4}).$$

Ex. 1. Find in degrees the values of X which satisfy the equation

$$\sin X = -\tfrac{1}{2}.$$

Here $\sin X = -\sin 30°$; $\therefore$ $X = -30°$ is the numerically smallest solution.

Hence the general solution is $X = n.180° - (-1)^n.30°$, as required.

We might also have derived the solution from the positive value $X = 210°$, giving $X = n.180° + (-1)^n.210°$. Both formulae represent the same series of angles, but the values of n corresponding to any given angle are different. Thus the solution $X = 330°$ is got by putting $n = 2$ in the first formula and $n = 3$ in the second.

Ex. 2. Find in radians the general solution of the equation

$$\cos\theta = -\tfrac{1}{2}\sqrt{2}.$$

Here $\cos\theta = -\cos\frac{1}{4}\pi$; $\therefore$ $\theta = \pi - \frac{1}{4}\pi = \frac{3}{4}\pi$ is the smallest positive solution, and is taken as the principal value, since the smallest negative solution $(-\frac{3}{4}\pi)$ is numerically equal to it.

Hence the general solution is $\theta = 2n\pi \pm \frac{3}{4}\pi$.

Ex. 3. Solve the equation

$$\sin\theta + \sqrt{3}.\cos\theta = 0.$$

Dividing by $\cos\theta$, we have

$$\tan\theta + \sqrt{3} = 0, \quad \text{or} \quad \tan\theta = -\sqrt{3} = -\tan\tfrac{1}{3}\pi;$$

$\therefore$ $\theta = -\frac{1}{3}\pi$ is the numerically smallest solution.

Hence the general solution is $\theta = n\pi - \frac{1}{3}\pi$.

Since $\theta = +\frac{2}{3}\pi$ is also a solution, we might equally well write the general solution $n\pi + \frac{2}{3}\pi$, and this may be got by writing $n+1$ for n in the first solution.

109. We conclude this chapter with an example of a slightly different form of equation, that may be solved by means of the general forms of § 107.

We give two different solutions, which lead to apparently different general expressions for the solutions. But if the angles are represented in a figure, it will be found that both expressions represent the same set of angles. This point is a very important one. If in solving any trigonometric equation the student should obtain a totally different general expression to the answer given, both results may be perfectly correct, and the only means of testing this is to see whether the angles represented by both formulae are the same. This can generally best be done by means of a figure.

Ex. Solve (in degrees) the equation $\sin X = \cos 2X$.

First solution— $\qquad \sin X = \sin(90° - 2X)$;

$$\therefore \quad X = n.180° + (-1)^n (90° - 2X);$$

$$\therefore \quad X + (-1)^n (2X) = n.180° + (-1)^n 90°,$$

$$\therefore X = \frac{n.180° + (-1)^n.90°}{1 + (-1)^n.2}.$$

Second solution— $\qquad \cos(90° - X) = \cos 2X$;

$$\therefore \quad 90° - X = n.360° \pm 2X;$$

$$\therefore \quad X = \frac{90° - n.360°}{\pm 2 + 1}.$$

Consider the angles represented by the first solution.

Put $n = 2m$ when n is even, $n = 2m+1$ when n is odd. The two cases give respectively

$X = m.120° + 30°$,

$X = -(2m+1)\,180° + 90° = -90° - m.360°$,

of which the first represents 30°, 150°, 270°, and angles coterminal with them, and the second similarly represents $-90°$ and coterminal angles (Fig. 90). These are, however, coterminal with 270°, and therefore are *also* represented by the first solution.

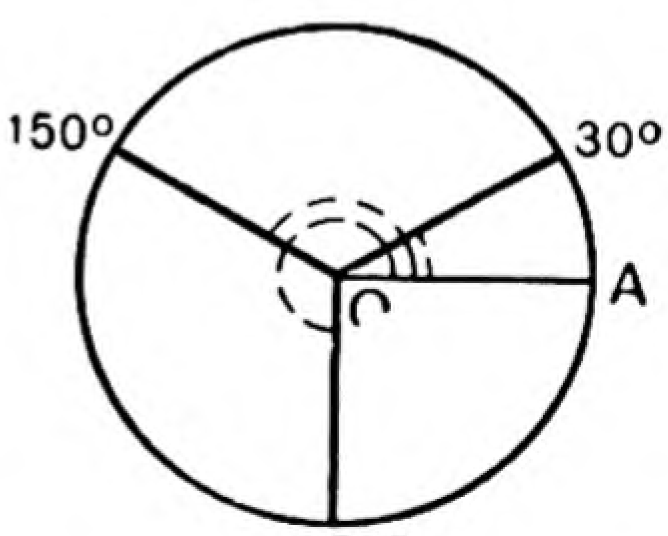


Fig. 90.

Taking the second solution, we have corresponding to the upper and lower signs

$$X = 30^\circ - n.120^\circ,$$
$$X = -90^\circ + n.360^\circ,$$

which evidently represent the same angles as before, n corresponding to $-m$ of the first solution.

EXAMPLES IX.

1. Prove that

$$\cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cot^{-1}\sqrt{\frac{a-x}{x-b}}.$$

2. Prove that $\sin^{-1}(\cos x) + \cos^{-1}(\sin y) + x + y = \pi$.

3. Find $\tan(\sec^{-1} x)$, and prove that

$$\sin[\cos^{-1}\{\tan(\sec^{-1} x)\}] = \sqrt{2-x^2}.$$

4. Are $\sin^{-1}\left(\tan\frac{\pi}{4}\right)$ and $\tan\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$ the same?

5. If $\sin^{-1} m + \sin^{-1} n = \frac{\pi}{2}$, show that

$$\sin^{-1} m = \cos^{-1} n.$$

6. Find the values of

$$\tan\left[\sin^{-1}\left\{\cot\left(\sec^{-1}\frac{1}{x}\right)\right\}\right],$$
$$\sin[\cos^{-1}[\sin[\cos^{-1}[\sin\{\cos^{-1}\sqrt{(1-a^2)}\}]]]],$$

and $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$.

7. Find x from the equation

$$\sin(\cot^{-1}\tfrac{1}{2}) = \tan(\cos^{-1}\sqrt{x}).$$

8. The sine of an unknown angle θ being given, equal to $\sin \alpha$, where α is given, investigate a general expression for the angle θ.

Write down in one formula all the angles which have $\frac{1}{2}$ for their sine.

9. Find an expression for all the angles which have the same tangent as a given angle A.

10. What is the value of θ which satisfies the equations

$$5\sin\theta + 3 = 0 \quad \text{and} \quad 5\cos\theta + 4 = 0\,?$$

11. If $\sin^2 A + \cos^2 B = 1$, what is the relation between A and B?

12. If $\cos 41^\circ\, 24'\, 34{\cdot}6'' = \frac{3}{4}$, find an angle θ which satisfies the equation $4\cos 2\theta + 3 = 0$.

13. Find the value or values of θ less than 180° which satisfy the equations (a) $2\cos\theta + 1 = 0$, (b) $\tan\theta + 1 = 0$.

14. If $2\tan 26^\circ\, 34' = 1$, write down all the values of θ less than four right angles for which $2\tan\theta + 1 = 0$.

15. Given $\sin\theta = \frac{1}{3}$, find a series of values of θ which satisfy the equation.

16. Find $\sin A$ from the equation $\tan A + \sec A = a$.

17. If $\cos A - \sin A = \frac{1}{5}\sqrt{5}$, find $\tan A$.

18. If $\tan A + \sec A = 2$, prove that $\sin A = \frac{3}{5}$, A being $< 90°$.

19. If $\sin A = \frac{3}{5}$, show that $\tan A - \sec A = -\frac{1}{2}$, when A is acute.

20. If $\tan A + \sec A = 3$, show that $\sin A = \frac{4}{5}$, A being acute.

Solve the following equations (21–29):—

21. $\tan\theta = 2\sin\theta$.

22. $\sin^2\theta + \cos^2(90° - \theta) = 1$.

23. $\sin\theta + 2\cos\pi + 4\tan\dfrac{\pi}{4} = 1$.

24. $\operatorname{cosec} x = 2\sin x$.

25. $\sin 2x = \cos 3x$.

26. $\sin 3\theta = \sin 4\theta$.

27. $\sin 5x = \cos 50°$.

28. $\cos\theta = \tan\phi$, $\cot\theta = \sin\phi$.

29. $\tan(A-B) = \dfrac{1}{\sqrt{3}}$, $\sin(A+B) = 1$.

30. Show that the series

$$(2n-1)\frac{\pi}{2} + (-1)^n\frac{\pi}{3}$$

represents exactly the same angles as the series

$$2n\pi \pm \frac{\pi}{6}.$$

31. Explain how it comes about that the same series of angles are indicated by the two equations

$$\theta + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{6} \quad \text{and} \quad \theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

32. If
$$\tan(2\alpha - 3\beta) = \cot(3\alpha - 2\beta),$$
and also
$$\tan(2\alpha + 3\beta) = \cot(3\alpha + 2\beta),$$
then both α and β are multiples of $\pi/10$.

CHAPTER X.

TRIGONOMETRIC EQUATIONS AND ELIMINATION.

110. **Equations involving only one function of the unknown angle.**—From the cases where one of the trigonometric functions of an angle is given, we naturally pass on to those in which some equation is given involving one or more of the trigonometric functions of an unknown angle, and it is required to find for what values of the angle the equation is satisfied. This process (as in Algebra) is called **solving the equation.**

When only one function of the unknown angle enters into an equation its value or values may be found by the ordinary rules of algebra, and the general expressions for the angle may be written down by the last chapter.

NOTE.—Where quadratic equations have to be solved in our illustrative examples, *the results alone will sometimes be given.* The reader is assumed to be familiar with the methods of solving quadratics in algebra, and is therefore left to supply the necessary intermediate steps. **This should be done in every case as an exercise.**

Ex. 1. Solve $\tan^2\theta+\frac{2}{3}\sqrt{3}\tan\theta-1=0.$

Treating this equation as a quadratic in $\tan\theta$, we have

$$\tan^2\theta+2\left(\tfrac{1}{3}\sqrt{3}\right)\tan\theta+\left(\tfrac{1}{3}\sqrt{3}\right)^2=1+\tfrac{1}{3}=\tfrac{4}{3}=\tfrac{4}{9}\times 3;$$

$$\therefore\quad \tan\theta+\tfrac{1}{3}\sqrt{3}=\pm\tfrac{2}{3}\sqrt{3};$$

$$\therefore\quad \tan\theta=\tfrac{1}{3}\sqrt{3},\quad\text{or}\quad -\sqrt{3}.$$

The principal values of θ are $30°$ and $-60°$;

$$\therefore\quad \theta=n.180°+30°,\ \text{or}\ n.180°-60°,$$

i.e.
$$\theta=n\pi+\tfrac{1}{6}\pi,\ \text{or}\ n\pi-\tfrac{1}{3}\pi.$$

Ex. 2. Solve $2\cos^2\theta-3\cos\theta-2=0.$

By solving this quadratic, *the student will find* $\cos\theta=-\frac{1}{2}$ or $\cos\theta=2$. But the latter solution is impossible. Hence, the only admissible solutions are given by $\theta=2n\pi\pm\frac{2}{3}\pi$.

111. Equations involving more than one function of the unknown angle. (*First method.*)—In solving equations of the present class, some such procedure as the following will often be found the best to adopt.

1st. Express all trigonometric functions of the unknown angle in terms of one of them—usually either the sine, cosine, or tangent, choosing this function so as to avoid introducing radicals if possible.

2nd. If the unknown quantity occurs under a radical, and this cannot be avoided by *any* choice of the function in the first process, transpose the radical to one side of the equation and square (just as in solving an "equation involving surds" in Algebra).

3rd. Solve the equation cleared of surds as an ordinary quadratic.

4th. If any solutions make the sine or cosine numerically greater than 1 or the cosecant or secant numerically less than 1, reject them as impossible.

5th. If the equation has been rationalised by squaring, substitute in the original equation, and thus determine one of the *other* trigonometric functions of the angle.

6th. Write down (by the last chapter) the general expression for all the angles whose functions have the required values.

Ex. 1. Solve $$\sin\theta - \cos\theta = \sqrt{2},$$

$$\sin\theta - \sqrt{2} = \cos\theta = \sqrt{(1-\sin^2\theta)}.$$

Square; $$\sin^2\theta - 2\sqrt{2}\sin\theta + 2 = 1 - \sin^2\theta,$$

$$2\sin^2\theta - 2\sqrt{2}\sin\theta + 1 = 0,$$

$$(\sqrt{2}\sin\theta - 1)^2 = 0,$$

$$\sin\theta = \frac{1}{\sqrt{2}},\quad \theta = \sin^{-1}\frac{1}{\sqrt{2}} = 45^\circ,\quad 135^\circ, \text{ etc.}$$

Substitute to find the sign of $\cos\theta$; $\cos\theta = \sin\theta - \sqrt{2} = -\dfrac{1}{\sqrt{2}}$;

$$\therefore\quad \theta \text{ cannot} = 45^\circ, \text{ but} = 135^\circ.$$

135° is the smallest angle which satisfies the equation.

Also, since $\sin\theta = \sin 135^\circ$ and $\cos\theta = \cos 135^\circ$;

$$\therefore\quad \theta = n.360^\circ + 135^\circ \text{ (by § 108)};$$

or, in circular measure, $$\theta = 2n\pi + \frac{3\pi}{4}.$$

Ex. 2. Solve $\sin\theta+\cos\theta = \sqrt{2}$.

Following the steps of Ex. 1, we shall arrive at the same equation

$$(\sqrt{2}\sin\theta-1)^2 = 0;$$

$$\therefore\quad \sin\theta = \frac{1}{\sqrt{2}},\quad \theta = \sin^{-1}\frac{1}{\sqrt{2}} = 45^\circ,\quad 135^\circ,\quad \text{and other values.}$$

Substitute to find the sign of $\cos\theta$;

$$\cos\theta = \sqrt{2}-\sin\theta = \frac{1}{\sqrt{2}};$$

$$\therefore\quad \theta \text{ cannot} = 135^\circ,\ \text{but} = 45^\circ;$$

$\therefore$ 45° or $\frac{1}{4}\pi$ is the smallest angle which satisfies the equation; and, since $\cos\theta = \cos\frac{1}{4}\pi$ and $\sin\theta = \sin\frac{1}{4}\pi$, the angle θ must be coterminal with $\frac{1}{4}\pi$, so that the general solution in radians is

$$\theta = 2n\pi + \frac{\pi}{4}.$$

Ex. 3. Solve $\tan\theta+\cot\theta = 4$.

Taking $\tan\theta$ as the unknown quantity, we have

$$\tan\theta + \frac{1}{\tan\theta} = 4.$$

Clearing of fractions,

$$\tan^2\theta+1 = 4\tan\theta;\qquad \therefore\quad \tan^2\theta-4\tan\theta+1 = 0.$$

"Completing the square,"

$$\tan^2\theta-4\tan\theta+4 = 3,\quad \text{or}\quad (\tan\theta-2)^2 = 3;$$

$$\therefore\quad \tan\theta-2 = \pm\sqrt{3},\quad \text{or}\quad \tan\theta = 2\pm\sqrt{3},$$

also $$\cot\theta = 4-\tan\theta;\qquad \therefore\quad \cot\theta = 2\mp\sqrt{3}.$$

In accordance with § 108, we write the general solutions

$$\theta = n\pi+\tan^{-1}(2+\sqrt{3}),\quad \text{or}\quad n\pi+\tan^{-1}(2-\sqrt{3}).$$

Ex. 4. Solve the equation

$$2\cot^2\theta-\cot\theta\operatorname{cosec}\theta-\operatorname{cosec}^2\theta = 0.$$

This may be written

$$\frac{2\cos^2\theta}{\sin^2\theta}-\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta}-\frac{1}{\sin^2\theta} = 0;$$

$$\therefore\quad 2\cos^2\theta-\cos\theta-1 = 0.$$

Solving as a quadratic in $\cos\theta$, we find

$$\cos\theta = \frac{1\pm\sqrt{(1+8)}}{4} = \frac{1\pm3}{4}$$

$$= 1,\ \text{or} -\tfrac{1}{2} = \cos 0,\quad \text{or}\quad \cos\tfrac{2}{3}\pi.$$

Hence the general solutions are

$$\theta = 2n\pi,\quad \text{or}\quad \theta = (2n\pm\tfrac{2}{3})\pi.$$

If the boundary lines of the various angles be represented as in Fig. 91, they will be found to diverge at angles of 120° with each other; hence the angles are all multiples of 120°. The general solution may therefore be also written in the form

$$\theta = \tfrac{2}{3}n\pi.$$

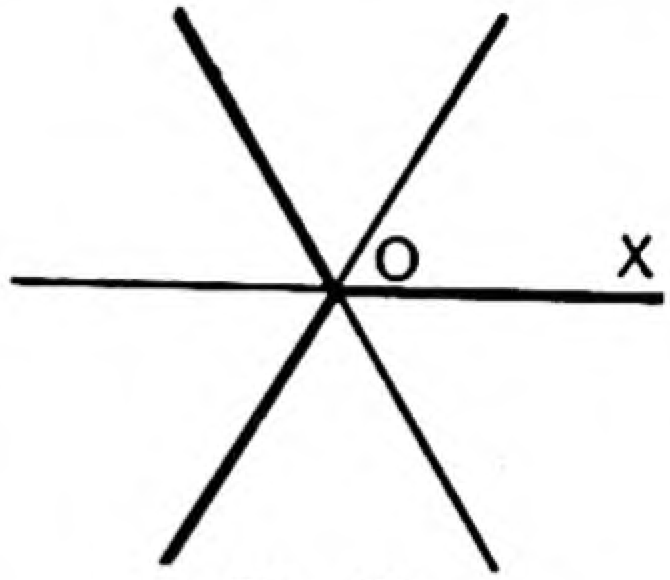


Fig. 91.

112. Equations involving more than one function. (*Second method.*)—It is sometimes more convenient to write down the identities connecting the different functions which occur in the given equation, and to regard these identities together with the given equation as *simultaneous equations*, by solving which the different functions can be determined.

Ex. 1. Solve $\cos\theta - \sin\theta = \frac{1}{2}$.

Let $\cos\theta = x$ and $\sin\theta = y$. Then the equation becomes

$$x - y = \tfrac{1}{2} \quad\text{.................(i)}$$

also the identity $\cos^2\theta + \sin^2\theta = 1$ becomes

$$x^2 + y^2 = 1 \quad\text{.................(ii)}$$

Solve (i) *and* (ii) *as simultaneous equations for x and y.*

Squaring (i), $x^2 - 2xy + y^2 = \frac{1}{4}$;

$\therefore$ by (ii), $2xy = \frac{3}{4}$; $\therefore$ $x^2 + 2xy + y^2 = 1\frac{3}{4} = \frac{7}{4}$;

Extract the square root;

$$\therefore \quad x + y = \pm\tfrac{1}{2}\sqrt{7} \quad\text{.................(iii)}$$

From (i) and (iii),

$$2x = \pm\tfrac{1}{2}\sqrt{7} + \tfrac{1}{2}, \quad 2y = \pm\tfrac{1}{2}\sqrt{7} - \tfrac{1}{2};$$

that is, $\cos\theta = \frac{1}{4}(\pm\sqrt{7} + 1)$, $\sin\theta = \frac{1}{4}(\pm\sqrt{7} - 1)$;

$\therefore$ either $\theta = 2n\pi + \cos^{-1}\frac{1}{4}(\sqrt{7}+1) = 2n\pi + \sin^{-1}\frac{1}{4}(\sqrt{7}-1)$.

or $\theta = 2n\pi + \cos^{-1}\frac{1}{4}(-\sqrt{7}+1) = 2n\pi + \sin^{-1}\frac{1}{4}(-\sqrt{7}-1)$,

Ex. 2. Solve $\sec\theta + \tan\theta = 4$ ....(i)

Taking the identity $\sec^2\theta - \tan^2\theta = 1$, and dividing by the given equation, we have $\sec\theta - \tan\theta = \frac{1}{4}$ ....(ii)

From (i) and (ii), $\sec\theta = 2\frac{1}{8}$, $\tan\theta = 1\frac{7}{8}$.

Hence also $\sin\theta = \tan\theta \div \sec\theta = \frac{15}{17}$, $\cos\theta = 1 \div \sec\theta = \frac{8}{17}$, etc.

Since *all* the trigonometric functions have known values, the general expression is $\theta = 2n\pi + \tan^{-1}\frac{15}{8}$.

113. **Homogeneous equations** in $\sin\theta$ and $\cos\theta$, *i.e.* equations of such forms as $a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta=0$ are at once reducible to equations for $\tan\theta$. To exemplify the method, we proceed to apply it to a slightly different form of equation.

Ex. $\operatorname{cosec}^2\theta-\frac{2}{3}\sqrt{3}\operatorname{cosec}\theta\sec\theta-\sec^2\theta=0$ ....(1)

We may write the equation

$$\frac{1}{\sin^2\theta}-\frac{2\sqrt{3}}{3\sin\theta\cos\theta}-\frac{1}{\cos^2\theta}=0 \qquad\text{..................(2)}$$

Clearing of fractions, we have

$$\cos^2\theta-\tfrac{2}{3}\sqrt{3}\sin\theta.\cos\theta-\sin^2\theta=0 \qquad\text{..................(3)}$$

a homogeneous equation in $\cos\theta$ and $\sin\theta$.

Dividing throughout by $\cos^2\theta$, we have

$$1-\tfrac{2}{3}\sqrt{3}\tan\theta-\tan^2\theta=0 \qquad\text{........................(4)}$$

an equation which might have been derived more quickly by simply multiplying (2) by $\sin^2\theta$.

Solving as a quadratic in $\tan\theta$, *the student will find*

$$\tan\theta=-\sqrt{3} \text{ or } \tfrac{1}{3}\sqrt{3}=\tan(-60^\circ) \text{ or } \tan 30^\circ,$$

i.e. $$\tan\theta=\tan(-\tfrac{1}{3}\pi) \text{ or } \tan\tfrac{1}{6}\pi.$$

Hence the general solutions in circular measure are

$$\theta=(n-\tfrac{1}{3})\pi \quad\text{and}\quad \theta=(n+\tfrac{1}{6})\pi.$$

114. **Elimination.**—If instead of a single equation, we have given *two* equations involving *one* unknown quantity, the values of the unknown quantity found by solving the first equation must satisfy the second. When these values have been substituted we obtain an equation from which the original unknown quantity is absent. This equation is called the **eliminant** of the two given equations, and the process of obtaining it is called **eliminating** the unknown quantity from the given equations.

Thus, if we are given the two equations

$$\sin\theta=x\cos\theta,\quad b=y\tan\theta,$$

we know from the first equation that $\tan\theta=x$; and, since this value must satisfy the second, we have on substitution

$$b=yx.$$

This equation does not involve θ, and therefore it is the *eliminant* of the two equations in θ.

In eliminating an unknown angle from two trigonometric equations, the identities of Chap. VII. frequently have to be used.

Ex. 1. Eliminate θ from

$$x = a \cos \theta \quad \text{and} \quad y = a \sin \theta.$$

The first gives
$$\cos \theta = \frac{x}{a},$$

and the second
$$\sin \theta = \frac{y}{a}.$$

But
$$\cos^2 \theta + \sin^2 \theta = 1;$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2} = 1, \quad \text{or} \quad x^2 + y^2 = a^2,$$

the eliminant required.

Ex. 2. Eliminate θ from

$$a \sec \theta - b \tan \theta = c \quad \text{.............................(i)}$$
$$a \sin \theta + b \cos \theta = c \quad \text{.............................(ii)}$$

Multiplying (i) throughout by $\cos \theta$, and transposing, we have

$$b \sin \theta + c \cos \theta = a \quad \text{.............................(iii)}$$

Solving (ii), (iii) for $\sin \theta$ and $\cos \theta$, we have

$$(ac - b^2) \sin \theta = c^2 - ab, \quad (ac - b^2) \cos \theta = a^2 - bc.$$

Substituting in $\sin^2 \theta + \cos^2 \theta = 1$, and clearing of fractions, we obtain

$$(c^2 - ab)^2 + (a^2 - bc)^2 = (ac - b^2)^2,$$

the eliminant required.

EXAMPLES X.

1. Write down the general values of θ satisfying

$$(a)\ 2 \cos \theta = \sqrt{2}, \qquad (b)\ \tan 2\theta = 1.$$

SOLVE the following equations (2-28):—

2. $\cos^2 \theta - 2 \cos \theta + 1 = 0.$
3. $3 \cos^2 x + 2\sqrt{3} \cos x = 5\frac{1}{4}.$
4. $\tan^4 x - 4 \tan^2 x + 3 = 0.$
5. $16 \sin^4 x - 16 \sin^2 x + 1 = 0.$
6. $8 \sin^4 \theta - 6 \sin^2 \theta + 1 = 0.$
7. $3 \tan^4 \theta - 10 \tan^2 \theta + 3 = 0.$
8. $\sec^4 x - 6 \sec^2 x + 8 = 0.$
9. $3 \sin \theta = 2 \cos^2 \theta.$
10. $\sin \theta = 1 - \cos \theta.$
11. $\operatorname{cosec} x = 2 \sin x.$
12. $1 + \cos x = \frac{3}{4} \sec x.$
13. $8 \cos^4 x + 10 \sin^2 x = 7.$
14. $\sin x + \cos x = 1.$
15. $8 \sin^2 \theta - 2 \cos \theta = 5.$
16. $2 \cos^2 \theta + 11 \sin \theta = 7.$
17. $2\sqrt{3} \cos^2 A = \sin A.$
18. $\cos^2 x - \sin x - \frac{1}{4} = 0.$
19. $\cos^2 x + 2 \sin x = \frac{7}{4}.$
20. $\cot x - \tan x = 2.$
21. $\tan x - \cot x = \tan \alpha - \cot \alpha.$
22. $\tan^2 \theta - \sec \theta = 1.$
23. $5 \tan^2 x - \sec^2 x = 11.$
24. $\tan \theta + \sec \theta = \sqrt{3}.$
25. $\sec x \operatorname{cosec} x - \tan x = 2.$

26. $\sin\theta+\operatorname{cosec}\theta=\frac{5}{2}$.

27. $\cos^2\theta+\sin\theta+\dfrac{3}{\operatorname{cosec}\theta}=\frac{11}{4}$.

28. $\cos^2 x+\cos^2 y=\frac{344}{225}$, $\sin x\sin y=\frac{1}{5}$.

29. Eliminate x from the equations

$$a\cos x+b\sin x=c,\quad b\cos x-a\sin x=d.$$

30. Eliminate x from the equations

$$\cot x=a,\quad \sec x=b.$$

31. Eliminate ϕ from the equations

$$\sec\phi=a,\quad \operatorname{cosec}\phi=b.$$

32. Eliminate θ from the equations

$$a\cos\theta+b\sin\theta=c,\quad b\cos\theta+c\sin\theta=a.$$

33. Eliminate θ from the equations

$$a\sec\theta-c\tan\theta=d,\quad b\sec\theta+d\tan\theta=c.$$

34. Eliminate θ between

$$\operatorname{cosec}\theta-\sin\theta=a,\quad \sec\theta-\cos\theta=b.$$

35. Eliminate θ between

$$\cos\theta-\sin\theta=a\quad\text{and}\quad\tan\theta=c\sec^2\theta.$$

36. Eliminate θ and ϕ between the equations

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2,\ \frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2,\quad \theta-\phi=\frac{\pi}{2}.$$

37. Eliminate ϕ from the equations

$$p\operatorname{cosec}\phi+q\cot\phi=r,\quad s\operatorname{cosec}\phi-r\cot\phi=q.$$

38. Eliminate ϕ from the equations

$$m\cos^2\phi+n\cos\phi=p,\quad m'\sec^2\phi+n'\sec\phi=p'.$$

39. Eliminate θ and ϕ from the equations

$$\cos\theta+\cos\phi=a,\quad \cot\theta+\cot\phi=b,\quad \operatorname{cosec}\theta+\operatorname{cosec}\phi=c.$$

40. Eliminate θ from the equations

$$a\tan^2\theta+b\tan\theta+c=0.\quad a'\cot^2\theta+b'\cot\theta+c'=0.$$

CHAPTER XI.

TRIGONOMETRIC FUNCTIONS OF A SUM OR DIFFERENCE.

115. In Chapter VIII. we proved certain relations between the trigonometric functions of such angles as $90°\pm A$ or $180°\pm A$ and those of A. In this chapter we shall express the sine, cosine, and tangent of the sum or difference of two angles, $A+B$ or $A-B$, in terms of functions of the component angles, A and B. The formulae which we shall prove are sometimes called the "**addition and subtraction formulae,**" because they enable us to find the functions of the angles formed by the *addition and subtraction* of two angles whose functions are known.

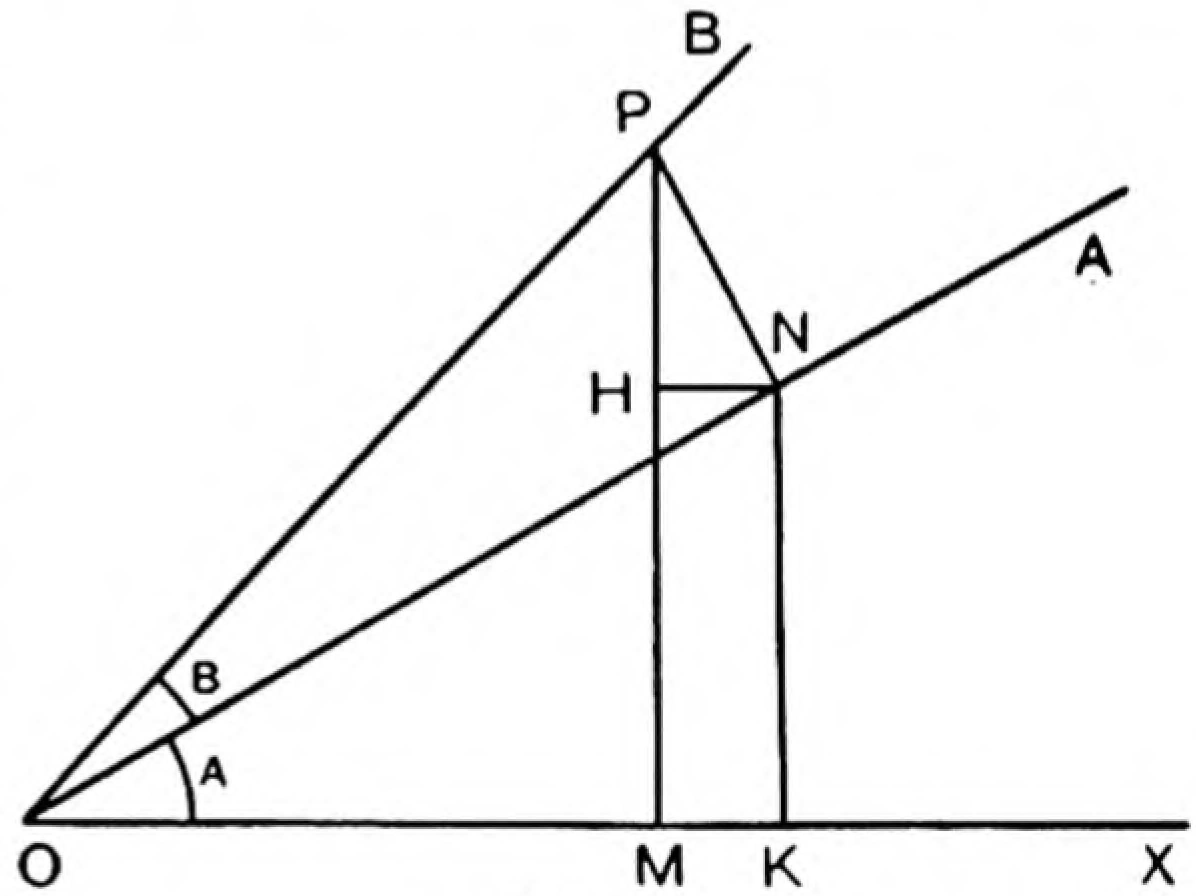


Fig. 92.

The student is recommended to pay especial attention to the proofs for cases in which all the angles (*i.e.* A, B, and the compound angle $A\pm B$) are positive and acute. In such cases we know that all the functions are positive, so that no difficulty arises as to the *signs* of the various lengths, and we need only consider their numerical magnitudes. But it is important to note that the **formulae proved in this chapter are true, whatever be the size of the angles involved, and whether these be positive or negative.** We shall give general proofs near the end of the chapter.

116. To prove the formulae

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \quad \ldots\ldots(47)$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \quad \ldots\ldots\ldots(48)$$

when all the angles are positive and acute.*

* This requires A, B to be positive angles, such that $A+B<90°$.

Let a radius vector revolving counter-clockwise describe the angle A in revolving from **OK** to **ON**, and let it subsequently describe an angle B in revolving from **ON** to **OP**.

Then the total angle described

$$\mathsf{KOP} = A+B.$$

Take any point **P** *in* **OP**, *the line bounding the compound angle* $A+B$.

Draw the perpendiculars **PM, PN** on **OK** and **ON.**

Draw the perpendiculars **NK, NH** on **OK** and **MP:**

Since **OMP, ONP** are right angles, a circle (on **OP** as diameter) will pass through **O, M, N, P,** and therefore (Euc. III. 21)

$$\angle \mathsf{HPN}, \text{ or } \angle \mathsf{MPN}, = \angle \mathsf{MON} = A;$$

$$\therefore \quad \frac{\mathsf{HP}}{\mathsf{NP}} = \cos A \text{ and } \frac{\mathsf{HN}}{\mathsf{NP}} = \sin A.$$

By definition,

$$\sin(A+B) = \frac{\mathsf{MP}}{\mathsf{OP}} = \frac{\mathsf{MH}+\mathsf{HP}}{\mathsf{OP}} = \frac{\mathsf{KN}}{\mathsf{OP}} + \frac{\mathsf{HP}}{\mathsf{OP}}.$$

Now $$\frac{\mathsf{KN}}{\mathsf{OP}} = \frac{\mathsf{KN}}{\mathsf{ON}} \cdot \frac{\mathsf{ON}}{\mathsf{OP}} = \sin A \cdot \cos B,$$

$$\frac{\mathsf{HP}}{\mathsf{OP}} = \frac{\mathsf{HP}}{\mathsf{NP}} \cdot \frac{\mathsf{NP}}{\mathsf{OP}} = \cos A \cdot \sin B;$$

$$\therefore \quad \sin(A+B) = \sin A \cos B + \cos A \sin B \quad \ldots\ldots(47)$$

Again,

$$\cos(A+B) = \frac{\mathsf{OM}}{\mathsf{OP}} = \frac{\mathsf{OK}-\mathsf{MK}}{\mathsf{OP}} = \frac{\mathsf{OK}}{\mathsf{OP}} - \frac{\mathsf{HN}}{\mathsf{OP}}.$$

Now $$\frac{\mathsf{OK}}{\mathsf{OP}} = \frac{\mathsf{OK}}{\mathsf{ON}} \cdot \frac{\mathsf{ON}}{\mathsf{OP}} = \cos A \cdot \cos B,$$

$$\frac{\mathsf{HN}}{\mathsf{OP}} = \frac{\mathsf{HN}}{\mathsf{NP}} \cdot \frac{\mathsf{NP}}{\mathsf{OP}} = \sin A \cdot \sin B;$$

$$\therefore \quad \cos(A+B) = \cos A \cos B - \sin A \sin B \quad \ldots\ldots(48)$$

ILLUSTRATIVE EXERCISE.

Write out the proof, using different letters for the angles, *e.g.* obtain the formulae for sin $(\theta+\phi)$ and cos $(\theta+\phi)$.

117. To find the sine and cosine of 75°.

$$\sin 75° = \sin (45°+30°)$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \ldots\ldots\ldots\ldots\ldots(51)$$
$$\cos 75° = \cos (45°+30°)$$
$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \ldots\ldots\ldots\ldots(52)$$

It is, however, best at first not to remember the values of sin 75° and cos 75°, but to work them out from the beginning and to do the same for 15°.

118. To prove the formulae

$$\sin (A-B) = \sin A \cos B - \cos A \sin B \quad \ldots\ldots\ldots(49)$$
$$\cos (A-B) = \cos A \cos B + \sin A \sin B \quad \ldots\ldots\ldots(50)$$

when all the angles are positive and acute.*

Let a radius vector describe the angle A in revolving counter-clockwise from **OM** to **ON**, and let it subsequently revolve clockwise (*i.e.* in the negative direction) through an angle of magnitude B in turning from **ON** to **OP**.

Then the angle **MOP** described between **OM** and **OP** (the initial and final positions)

$$= A - B.$$

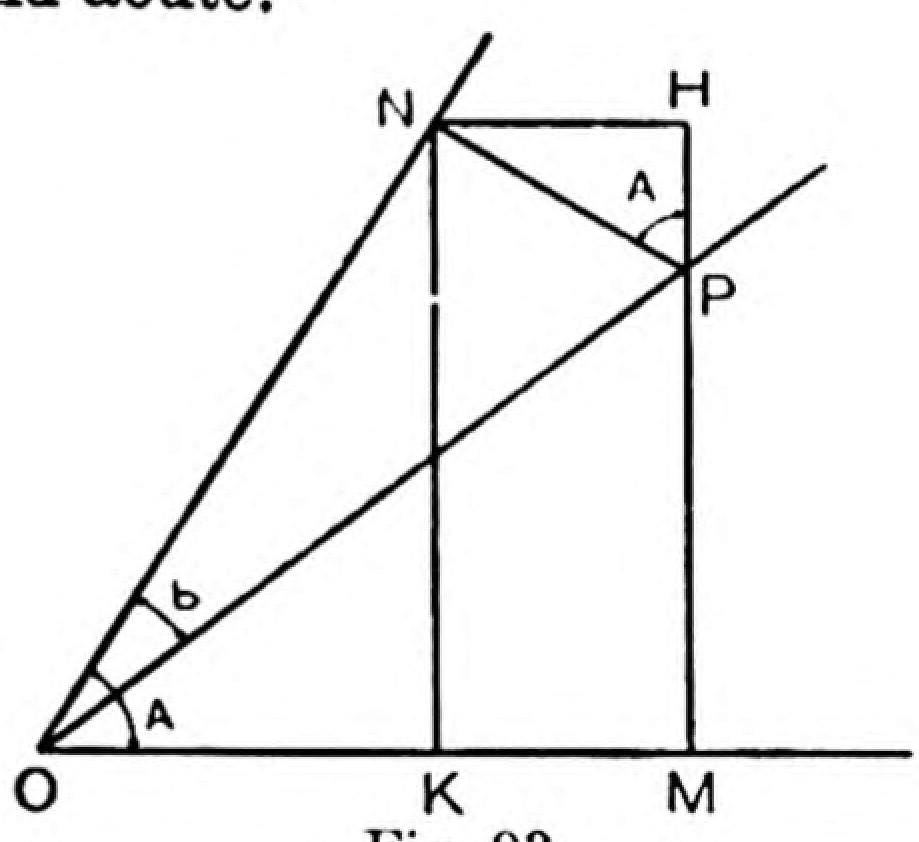


Fig. 93.

Take any point **P** *on the line bounding the compound angle* $A-B$.

Draw the perpendiculars **PM, PN** on **OM** and **ON**.

Draw the perpendiculars **NK, NH** on **OM** and **MP** (produced).

* This requires A, B to be positive acute angles such that $A > B$.

Since **OMP, ONP** are right angles, a circle (on **OP** as diameter) will pass through **OMNP**, and therefore (Euc. III. 22)

$$\angle \mathbf{HPN} = 180° - \angle \mathbf{MPN} = \angle \mathbf{MON} = A;$$

$$\therefore \quad \frac{\mathbf{PH}}{\mathbf{PN}} = \cos A \quad \text{and} \quad \frac{\mathbf{NH}}{\mathbf{NP}} = \sin A.$$

$$\sin(A-B) = \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{\mathbf{MH}-\mathbf{PH}}{\mathbf{OP}} = \frac{\mathbf{KN}}{\mathbf{OP}} - \frac{\mathbf{PH}}{\mathbf{OP}}.$$

Now $$\frac{\mathbf{KN}}{\mathbf{OP}} = \frac{\mathbf{KN}}{\mathbf{ON}} \cdot \frac{\mathbf{ON}}{\mathbf{OP}} = \sin A \,.\cos B,$$

$$\frac{\mathbf{PH}}{\mathbf{OP}} = \frac{\mathbf{PH}}{\mathbf{NP}} \cdot \frac{\mathbf{NP}}{\mathbf{OP}} = \cos A \,.\sin B;$$

$$\sin(\boldsymbol{A}-\boldsymbol{B}) = \sin \boldsymbol{A} \cos \boldsymbol{B} - \cos \boldsymbol{A} \sin \boldsymbol{B} \ldots\ldots \textbf{(49)}$$

Again,

$$\cos(A-B) = \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{\mathbf{OK}+\mathbf{KM}}{\mathbf{OP}} = \frac{\mathbf{OK}}{\mathbf{OP}} + \frac{\mathbf{NH}}{\mathbf{OP}}.$$

Now $$\frac{\mathbf{OK}}{\mathbf{OP}} = \frac{\mathbf{OK}}{\mathbf{ON}} \cdot \frac{\mathbf{ON}}{\mathbf{OP}} = \cos A \,.\cos B,$$

$$\frac{\mathbf{NH}}{\mathbf{OP}} = \frac{\mathbf{NH}}{\mathbf{NP}} \cdot \frac{\mathbf{NP}}{\mathbf{OP}} = \sin A \,.\sin B;$$

$$\therefore \quad \cos(\boldsymbol{A}-\boldsymbol{B}) = \cos \boldsymbol{A} \cos \boldsymbol{B} + \sin \boldsymbol{A} \sin \boldsymbol{B} \ldots\ldots \textbf{(50)}$$

Illustrative Exercise.

Write out a proof of the formulae for $\sin(B-A)$ and $\cos(B-A)$, supposing B to be greater than A.

119. To find the sine and cosine of 15°.

Since $$15° = 45° - 30°,$$

$$\mathbf{\sin 15°} = \sin 45° \cos 30° - \cos 45° \sin 30° = \frac{\sqrt{3}-1}{2\sqrt{2}} \ldots \textbf{(53)}$$

$$\mathbf{\cos 15°} = \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{3}+1}{2\sqrt{2}} \ldots \textbf{(54)}$$

Notice that $\sin 15° = \cos 75°$ and $\cos 15° = \sin 75°$, as they should be, since 15° and 75° are complementary.*

ILLUSTRATIVE EXERCISE.

Verify that the formulae for $\sin(60°-45°)$ and $\cos(60°-45°)$ lead to the same values for sin 15° and cos 15°.

120. **Summary.**—We thus have the following formulae, which are exceedingly important and should be remembered:—

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad \ldots\ldots (47)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad \ldots\ldots (48)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad \ldots\ldots (49)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad \ldots\ldots (50)$$

These results may also be expressed in two formulae, thus:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad \ldots\ldots (\text{I.})$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad \ldots\ldots (\text{II.})$$

where $A \pm B$ is read "*A plus or minus B*" and $A \mp B$ is read "*A minus or plus B*." In applying these formulae, either the upper sign or the lower sign must be taken consistently throughout, the upper sign being, of course, + in $\pm$ and − in $\mp$.

It will be found better to remember all the four formulae separately.

They may be expressed in words in some such forms as the following:—

sin sum	= sum of (each sine × other cosine)
	= sin (1st $\angle$) cos (2nd $\angle$) + cos (1st $\angle$) sin (2nd $\angle$);
sin difference	= sin (1st $\angle$) cos (2nd $\angle$) − cos (1st $\angle$) sin (2nd $\angle$);
cos sum	= product of cosines − product of sines;
cos difference	= product of cosines + product of sines.

---

* If the general expression $\dfrac{\sqrt{3} \mp 1}{2\sqrt{2}}$ be remembered, it is easy to decide which sign to take in any case. For $\sin 75° > \sin 15°$; hence sin 75° must have the greater, and sin 15° the less, value. Also the cosine decreases as the angle increases; hence cos 75° must have the less, and cos 15° the greater, value.

Illustrative Exercises.

Write down the following functions expressed in terms of the sines and cosines of the component angles, substituting numerical values for those that are known:—

(1) $\cos(\theta-\phi)$; (2) $\sin(\beta+\alpha)$; (3) $\sin(y-x)$;
(4) $\cos(x+y)$; (5) $\sin(90°-A)$; (6) $\cos(\frac{1}{2}\pi+\alpha)$;
(7) $\sin(45°+A)$; (8) $\sin(\frac{1}{4}\pi-\theta)$ (9) $\cos(B-45°)$;
(10) $\cos(\frac{1}{4}\pi+\alpha)$; (11) $\sin(A-60°)$; (12) $\cos(\frac{1}{6}\pi+\gamma)$.

The following examples are also instructive—

Ex. 1. To express the sine and cosine of $A+B+C$ in terms of those of A, B, C.

$$\begin{aligned}\sin(A+B+C) &= \sin\{(A+B)+C\}\\ &= \sin(A+B)\cos C+\cos(A+B)\sin C\\ &= (\sin A\cos B+\cos A\sin B)\cos C\\ &\qquad +(\cos A\cos B-\sin A\sin B)\sin C\\ &= \sin A\cos B\cos C+\sin B\cos C\cos A\\ &\qquad +\sin C\cos A\cos B-\sin A\sin B\sin C\end{aligned}$$

= the sum of three products formed of one sine and two cosines *minus* the product of all three sines.

$$\begin{aligned}\cos(A+B+C) &= \cos(A+B)\cos C-\sin(A+B)\sin C\\ &= (\cos A\cos B-\sin A\sin B)\cos C\\ &\qquad -(\sin A\cos B+\cos A\sin B)\sin C\\ &= \cos A\cos B\cos C-\cos A\sin B\sin C\\ &\qquad -\cos B\sin C\sin A-\cos C\sin A\sin B\end{aligned}$$

= the product of all three cosines *minus* the three products formed of one cosine and two sines.

Ex. 2. To express $\sec(A+B)$ and $\operatorname{cosec}(A+B)$ in terms of the secants and cosecants of A and B.

$$\sec(A+B)=\frac{1}{\cos(A+B)}=\frac{1}{\cos A\cos B-\sin A\sin B}.$$

Multiplying the numerator and denominator by

$$\sec A\sec B\operatorname{cosec} A\operatorname{cosec} B,$$

we have

$$\sec(A+B)=\frac{\sec A\sec B\operatorname{cosec} A\operatorname{cosec} B}{\operatorname{cosec} A\operatorname{cosec} B-\sec A\sec B}.$$

In like manner,

$$\operatorname{cosec}(A+B)=\frac{1}{\sin(A+B)}=\frac{1}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\sec A\sec B\operatorname{cosec} A\operatorname{cosec} B}{\sec A\operatorname{cosec} B+\operatorname{cosec} A\sec B}.$$

ILLUSTRATIVE EXERCISE.

Obtain corresponding formulae for the secant and cosecant of $\theta-\phi$.

121. Converse use of the $A \pm B$ formulae.—Writing the four fundamental formulae backwards thus:

$$\sin A \cos B+\cos A \sin B = \sin (A+B),$$
$$\sin A \cos B-\cos A \cos B = \sin (A-B),$$
$$\cos A \cos B-\sin A \sin B = \cos (A+B),$$
$$\cos A \cos B+\sin A \sin B = \cos (A-B),$$

we notice that they enable us to simplify any expression of form (product of two functions of two angles)

$\pm$ (product of remaining two functions),

the functions being in each case sines or cosines.

Ex. 1. Simplify $\cos (A+B) \cos B+\sin (A+B) \sin B$.

The expression evidently $= \cos (A+B-B) = \cos A$.

Ex. 2. Prove that $\tan A+\tan B = \dfrac{\sin (A+B)}{\cos A \cos B}$.

$$\tan A+\tan B = \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}=\frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B}$$
$$= \frac{\sin (A+B)}{\cos A \cos B}.$$

Ex. 3. Simplify (i) $\sin A-\tan B \cos A$ and (ii) $\sin A-\cot B \cos A$.

(i) $\sin A - \dfrac{\sin B}{\cos B} \cos A = \dfrac{\sin A \cos B-\cos A \sin B}{\cos B} = \dfrac{\sin (A-B)}{\cos B}$.

(ii) $\sin A - \dfrac{\cos B}{\sin B} \cos A = \dfrac{\sin A \sin B-\cos A \cos B}{\sin B} = -\dfrac{\cos (A+B)}{\sin B}$.

ILLUSTRATIVE EXERCISES.

Simplify (1) $\sin (B-A) \cos A+\cos (B-A) \sin A$;

(2) $(\tan A-\tan B) \cos A \cos B$; (3) $(\sin A+\tan B \cos A) \cos B$.

122. To express $\tan (A+B)$ and $\tan (A-B)$ in terms of $\tan A$ and $\tan B$.

$$\tan (A+B) = \frac{\sin (A+B)}{\cos (A+B)} = \frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B-\sin A \sin B}.$$

Dividing numerator and denominator by $\cos A \cos B$, we obtain

$$\frac{\sin A/\cos A+\sin B/\cos B}{1-\sin A \sin B/(\cos A \cos B)};$$

$$\therefore\quad \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad\text{.........}\quad (55)$$

In like manner,

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B};$$

$$\therefore\quad \tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B} \quad\text{...........}\quad (56)$$

Results (55) and (56) should be known thoroughly; they are of frequent use in proving identities. They may be combined into one formula, thus:

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B},$$

provided that either the upper or lower sign is taken consistently throughout.

Expressed in words the formulae become

$$\text{tan sum}=\frac{\text{sum of tangents}}{1-\text{product of tangents}};$$

$$\text{tan difference}=\frac{\text{difference of tangents}}{1+\text{product of tangents}}.$$

ILLUSTRATIVE EXERCISES.

Write down by the formulae, substituting numerical values where known:—

(1) $\tan(B-A)$; (2) $\tan(\theta+\phi)$; (3) $\tan(60^\circ+A)$;
(4) $\tan(a-\frac{1}{3}\pi)$; (5) $\tan(B-30^\circ)$; (6) $\tan(\frac{1}{6}\pi+a)$.

123. To find tan 75° and tan 15°.

$$\tan 75^\circ=\tan(45^\circ+30^\circ)=\frac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1+\dfrac{1}{\sqrt{3}}}{1-1\cdot\dfrac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Rationalising the denominator, we have

$$\tan 75^\circ = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2}$$

$$= 2+\sqrt{3} \quad \ldots\ldots\ldots\ldots \quad (57)$$

Similarly,

$$\tan 15^\circ = \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \text{ (on simplification)}$$

$$= 2-\sqrt{3} \quad \ldots\ldots\ldots\ldots \quad (58)$$

ILLUSTRATIVE EXERCISES.

(1) Find $\tan 15^\circ$ by the formula for $\tan(60^\circ - 45^\circ)$.

(2) Find $\tan 75^\circ$ from the fact that $\tan(90 - 15^\circ) = \cot 15^\circ$.

124. To express $\tan(45^\circ + A)$ and $\tan(45^\circ - A)$ in terms of $\tan A$

Since $\tan 45^\circ = 1$, therefore, by § 122,

$$\tan(45^\circ + A) = \frac{1+\tan A}{1-\tan A} \quad \ldots\ldots\ldots\ldots (59)$$

$$\tan(45^\circ - A) = \frac{1-\tan A}{1+\tan A} \quad \ldots\ldots\ldots\ldots (60)$$

These results are often useful in proving identities, so that it may, perhaps, be worth while to remember them.

The following are also instructive:—

Ex. 1. To express $\cot(A+B)$ and $\cot(A-B)$ in terms of cotangents of A and B.

$$\cot(A+B) = \frac{1}{\tan(A+B)} = \frac{1-\tan A \tan B}{\tan A + \tan B}.$$

Multiplying numerator and denominator by $\cot A \cot B$,

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$

Similarly,

$$\cot(A-B) = \frac{1+\tan A \tan B}{\tan A - \tan B} = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

These formulae are rarely used and should not be remembered, as they can at once be deduced from the tangent formulae.

Ex. 2. Prove $\dfrac{\tan A - \tan B}{\cot A + \tan B} = \tan A \tan(A-B)$.

$$\frac{\tan A - \tan B}{\cot A + \tan B} = \frac{\tan A - \tan B}{\dfrac{1+\tan A \tan B}{\tan A}} = \tan A \, \frac{\tan A - \tan B}{1+\tan A \tan B}$$

$$= \tan A \tan(A-B), \text{ from above.}$$

In like manner, it may be proved that

$$\frac{\tan A+\tan B}{\cot A-\tan B}=\tan A\tan(A+B).$$

We shall now give geometrical proofs of the results of § 122.

125. To prove geometrically that

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B} \quad \text{...........(55)}$$

when all the angles are positive and acute.

Take Fig. 94. As in § 116, prove that

$$\angle \mathsf{MPN}=A.$$

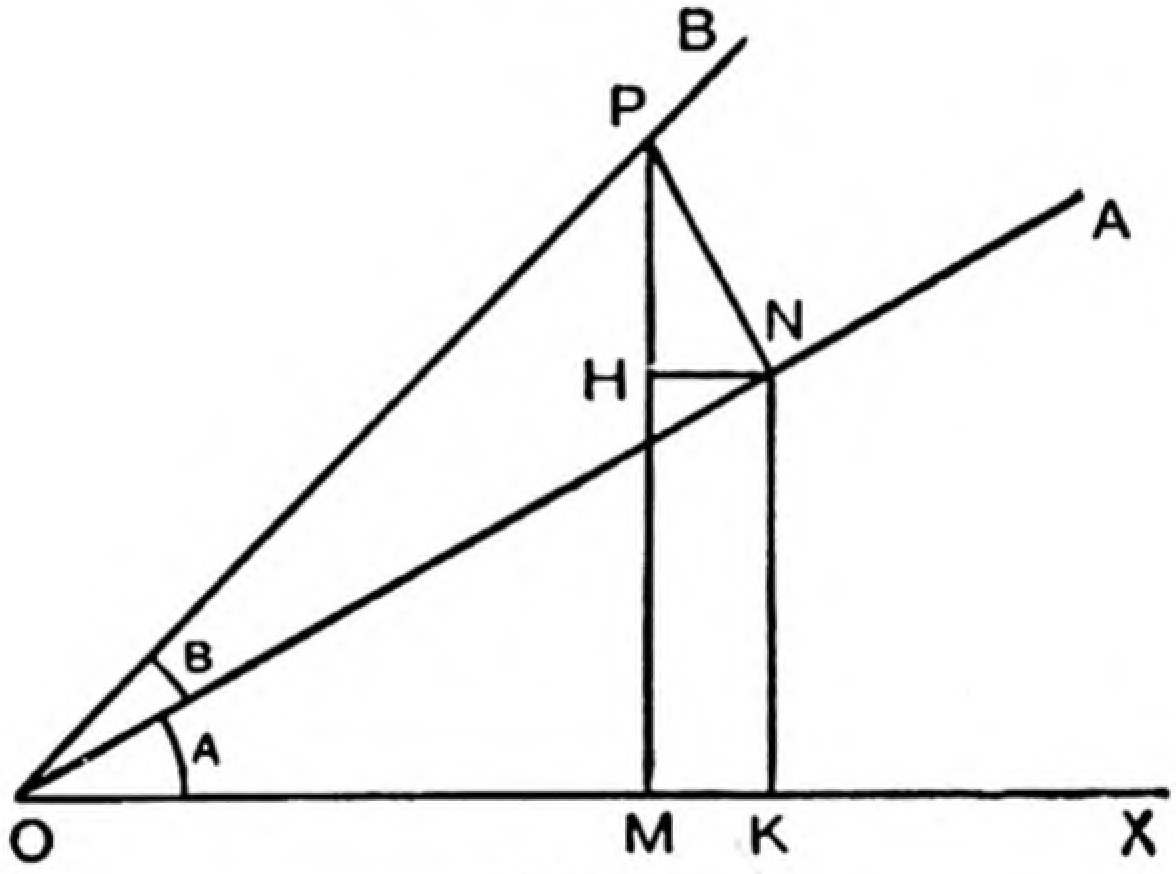


Fig. 94.

Then $$\tan(A+B)=\frac{\mathsf{MP}}{\mathsf{OM}}=\frac{\mathsf{KN}+\mathsf{HP}}{\mathsf{OK}-\mathsf{HN}}$$

$$=\frac{\dfrac{\mathsf{KN}}{\mathsf{OK}}+\dfrac{\mathsf{HP}}{\mathsf{OK}}}{1-\dfrac{\mathsf{HN}}{\mathsf{HP}}\cdot\dfrac{\mathsf{HP}}{\mathsf{OK}}}.$$

But $\triangle$s **HPN, NOK** are similar;

$$\therefore\quad \frac{\mathsf{HP}}{\mathsf{NP}}=\frac{\mathsf{OK}}{\mathsf{ON}};$$

$$\therefore\quad \frac{\mathsf{HP}}{\mathsf{OK}}=\frac{\mathsf{NP}}{\mathsf{ON}}=\tan B;$$

and $$\frac{\mathsf{KN}}{\mathsf{OK}} = \tan A, \quad \frac{\mathsf{HN}}{\mathsf{PH}} = \tan A;$$

$$\therefore \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{.............(55)}$$

126. To prove geometrically that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{........... (56)}$$

when all the angles are positive and acute.

Take Fig. 95, and proceed in exactly the same way as in § 118, to prove that

$$\angle \mathsf{HPN} = A.$$

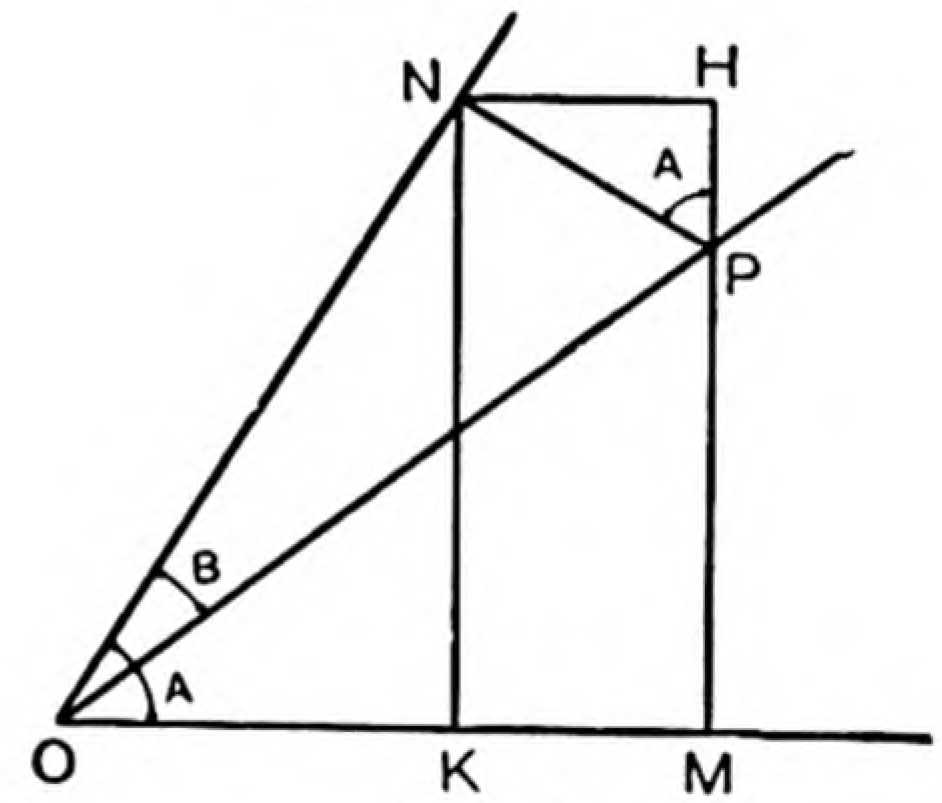


Fig. 95.

Then $$\tan(A-B) = \frac{\mathsf{MP}}{\mathsf{OM}} = \frac{\mathsf{KN}-\mathsf{PH}}{\mathsf{OK}+\mathsf{NH}}$$

$$= \frac{\dfrac{\mathsf{KN}}{\mathsf{OK}} - \dfrac{\mathsf{PH}}{\mathsf{OK}}}{1 + \dfrac{\mathsf{NH}}{\mathsf{HP}} \cdot \dfrac{\mathsf{HP}}{\mathsf{OK}}}.$$

But $\triangle$s **HPN, KON** are similar;

$$\therefore \quad \frac{\mathsf{HP}}{\mathsf{NP}} = \frac{\mathsf{OK}}{\mathsf{ON}};$$

$$\therefore \quad \frac{\mathsf{HP}}{\mathsf{OK}} = \frac{\mathsf{NP}}{\mathsf{ON}} = \tan B;$$

and $$\frac{KN}{OK} = \tan A, \quad \frac{HN}{PH} = \tan A;$$

$$\therefore \quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B} \quad \text{........(56)}$$

127. **Extension of the formulae to angles of any magnitude.** The formulae of the present chapter have hitherto only been proved to hold good when all the angles involved in them are positive and acute. They are, however, algebraically true for all angles, but the proofs are very difficult in all but the simplest cases owing to the complications introduced by the signs of the various lengths in the figure. A general proof will be given in the next article, but it may be of interest here to illustrate a method of extending them by means of the properties of related angles in the first quadrant.

[The student is advised not to attempt to read the generalised proofs of § 128 until the special proofs of §§ 116, 118, 125, 126, as well as the properties of functions of unrestricted angles, have been thoroughly mastered.]

Ex. 1. Find the values of sin 105° and cos 105°.

$$\sin 105° = \sin(180°-75°) = \sin 75°, \text{ by § 89,}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}, \text{ by § 117.}$$
$$\cos 105° = \cos(180°-75°) = -\cos 75°, \text{ by § 89.}$$
$$= -\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}, \text{ by § 117.}$$

Or, we might proceed thus:—

Assuming the formulae for $\sin(A+B)$ and $\cos(A+B)$ to be true when $A+B = 105°$, we have

$$\sin 105° = \sin(60°+45°)$$
$$= \sin 60° \cos 45° + \cos 60° \sin 45°$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$
$$\cos 105° = \cos(60°+45°)$$
$$= \cos 60° \cos 45° - \sin 60° \sin 45°$$
$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}.$$

Since $1 < \sqrt{3}$, cos 105° is negative, as it should be.

The agreement between the results of the two methods affords a verification that the "$A+B$" formulae are applicable in the present case.

Ex. 2. Assuming the formula for $\sin(A+B)$ to be true when A, B and $A+B$ are all $< 90°$, prove it to be true when A and B are $< 90°$, but $A+B > 90°$.

Here $A+B$ lies between $90°$ and $180°$; hence $180°-A-B$ lies between $0°$ and $90°$, and so also do $90°-A$ and $90°-B$. Hence, by assumption and by § 89,

$$\begin{aligned}\sin(A+B) &= \sin(180°-A-B) = \sin\{(90°-A)+(90°-B)\}\\ &= \sin(90°-A)\cos(90°-B)+\cos(90°-A)\sin(90°-B)\\ &= \cos A\sin B+\sin A\cos B. \qquad (\S\ 86)\end{aligned}$$

In this way the $A+B$ formulae can be proved to hold good if A and B are *any acute angles*.

Ex. 3. Assuming the formulae for the sine and cosine of $(A+B)$ true when A and B have any given values, to prove them to be true when either of these angles (say A) is increased by $90°$.

By § 91,

$$\begin{aligned}\sin(A+90°+B) &= \cos(A+B) = \cos A\cos B-\sin A\sin B\\ &= \sin(90°+A)\cos B+\cos(90°+A)\sin B, \qquad (\S\ 91)\end{aligned}$$

as was to be proved. Similarly for the cosine.

Since, by Ex. 2, the formulae hold if A and B are in the first quadrant, and since, by Ex. 3, they still hold if either A or B is increased by $90°$, it follows that they hold if A, B are either or both increased by any multiples of $90°$, and hence are any positive angles whatever. It may be similarly proved that the formulae hold when A, B are either or both decreased by $90°$, and the formulae may thus be extended to negative angles as well.

****128. To prove the formulae for the sine, cosine, and tangent of $(A+B)$ for angles of any magnitude whatever.**

Let a radius vector describe the angle A in revolving counter-clockwise from **OX** to **OA**, and let it subsequently revolve from **OA** to **OP** in the same direction, describing the angle B.

Then the total angle described between **OX** and **OP** $= A+B$.

Take a point **P** on the line **OP** bounding the compound angle $A+B$.

Draw the perpendiculars **PM**, **PN** on **OX** and **OA** (produced, if necessary).

Draw the perpendicular **NK** on **OX**.

Draw **N**x parallel to, and in the same sense as, **OX**, and let it meet **MP** (both being produced, if necessary) in **H**.

Let **Q** be any point in **NP**, or **NP** produced, such that $\angle$**ANQ** is a right angle described in the positive direction from **NA**.

The total angle x**NQ** (described in the positive direction from **N**x)

$$= x\mathbf{NA}+90° = \mathbf{XOA}+90° = A+90°;$$

$$\therefore \frac{\mathbf{NH}}{\mathbf{NP}} = \cos(A+90°) \quad \text{and} \quad \frac{\mathbf{HP}}{\mathbf{NP}} = \sin(A+90),$$

the positive directions of **NH**, **HP** being the same as the positive directions in defining the functions of the angles **XOA** and **XOP** at **O**; viz. left to right and upwards in the figure.

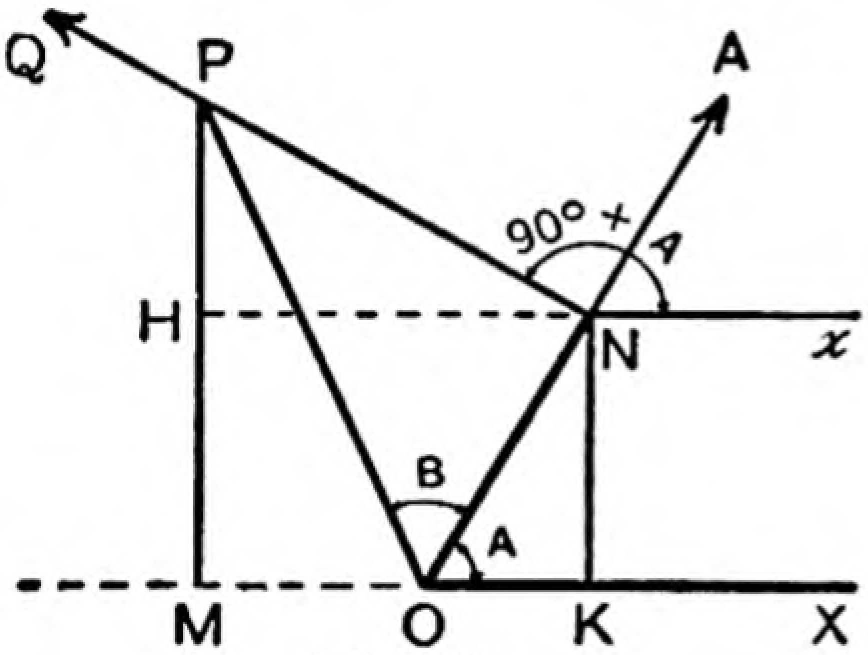


Fig. 96.

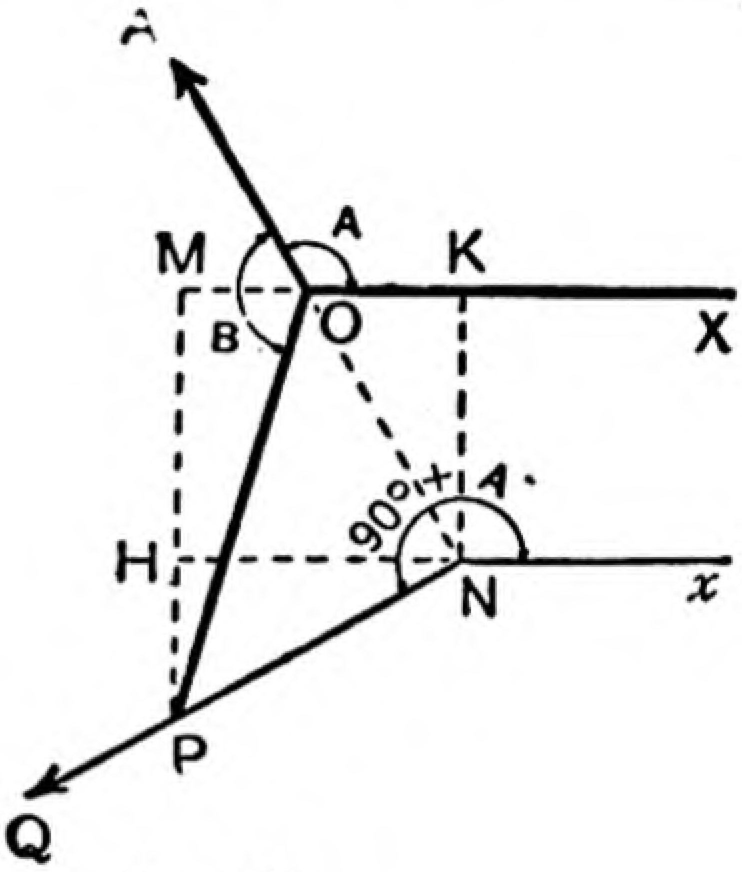


Fig. 97.

Again, $\dfrac{\mathbf{ON}}{\mathbf{OP}} = \cos B, \quad \dfrac{\mathbf{NP}}{\mathbf{OP}} = \sin B, \quad$ and $\quad \dfrac{\mathbf{NP}}{\mathbf{ON}} = \tan B.$

ON, **NP** are positive with reference to the angle B (*i.e.* **AOP**) when they lie along **OA**, **NQ**, respectively; negative when they lie along the produced directions of **AO**, **QN**, respectively.

Hence, if directions be represented by the order of the letters, we have, algebraically,

$$\sin(A+B) = \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{\mathbf{MH}+\mathbf{HP}}{\mathbf{OP}} = \frac{\mathbf{KN}}{\mathbf{OP}} + \frac{\mathbf{HP}}{\mathbf{OP}},$$

$$\frac{\mathbf{KN}}{\mathbf{OP}} = \frac{\mathbf{KN}}{\mathbf{ON}} \cdot \frac{\mathbf{ON}}{\mathbf{OP}} = \sin A \cos B,$$

$$\frac{\mathbf{HP}}{\mathbf{OP}} = \frac{\mathbf{HP}}{\mathbf{NP}} \cdot \frac{\mathbf{NP}}{\mathbf{OP}} = \sin(90°+A)\sin B = \cos A \sin B;$$

$$\therefore \quad \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

$$\cos(A+B) = \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{\mathbf{OK}+\mathbf{KM}}{\mathbf{OP}} = \frac{\mathbf{OK}}{\mathbf{OP}} + \frac{\mathbf{NH}}{\mathbf{OP}},$$

$$\frac{\mathsf{OK}}{\mathsf{OP}} = \frac{\mathsf{OK}}{\mathsf{ON}} \cdot \frac{\mathsf{ON}}{\mathsf{OP}} = \cos A \cos B,$$

$$\frac{\mathsf{NH}}{\mathsf{OP}} = \frac{\mathsf{NH}}{\mathsf{NP}} \cdot \frac{\mathsf{NP}}{\mathsf{OP}} = \cos (90^\circ + A) \sin B = -\sin A \sin B;$$

$$\therefore \quad \mathbf{\cos (A+B) = \cos A \cos B - \sin A \sin B.}$$

$$\tan (A+B) = \frac{\mathsf{MP}}{\mathsf{OM}} = \frac{\mathsf{MH} + \mathsf{HP}}{\mathsf{OK} + \mathsf{KM}} = \frac{\mathsf{KN} + \mathsf{HP}}{\mathsf{OK} + \mathsf{NH}}$$

$$= \frac{\dfrac{\mathsf{KN}}{\mathsf{OK}} + \dfrac{\mathsf{HP}}{\mathsf{OK}}}{1 + \dfrac{\mathsf{NH}}{\mathsf{HP}} \cdot \dfrac{\mathsf{HP}}{\mathsf{OK}}}.$$

Now $$\frac{\mathsf{HP}}{\mathsf{NP}} = \sin (90^\circ + A) = \cos A = \frac{\mathsf{OK}}{\mathsf{ON}}, \textit{ algebraically};$$

$$\therefore \quad \frac{\mathsf{HP}}{\mathsf{OK}} = \frac{\mathsf{NP}}{\mathsf{ON}} = \tan B, \quad \frac{\mathsf{KN}}{\mathsf{OK}} = \tan A,$$

and $$\frac{\mathsf{NH}}{\mathsf{HP}} = \cot (90^\circ + A) = -\tan A;$$

$$\mathbf{\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.}$$

[NOTE.—When B is obtuse (Fig. 97), **N** lies on the produced direction of **AO**, and not on the portion which bounds the angle A measured from **OX**. In this case **ON** is to be regarded as a *negative length* on the radius vector **OA** bounding the angle A, so that we still have, algebraically,

$$\sin A = \mathsf{KN}/\mathsf{ON}, \quad \cos A = \mathsf{OK}/\mathsf{ON}, \quad \tan A = \mathsf{KN}/\mathsf{OK},$$

and the proof holds good as in other cases.

A similar convention is, of course, required if **P** is on the side of **N** remote from **O**, in connection with the functions of the angle x**NP**, or $90^\circ + A$, at **N**.

The student should (at any rate, at first) consider exclusively those cases in which *the angle B lies between* 0 *and* 90°; these difficulties will not then occur.]

ILLUSTRATIVE EXERCISES.

Draw the figure and go through the proofs in the following cases:—

(1) When A is obtuse and B acute, and $A+B > 90^\circ$.

(2) When A is a negative and B a positive acute angle.

(3) When A lies between 90° and 180°, B between 0° and 90°, and $A+B$ between 180° and 270°.

****129. To prove the formulae for the sine, cosine, and tangent of $A-B$ for angles of any magnitude whatever.**

Let $\angle$ **XOA** (described counter-clockwise) $= A$,

$\angle$ **AOB** (described *clockwise*) $= B$;

then angle between **OX** and **OB** $= A-B$.

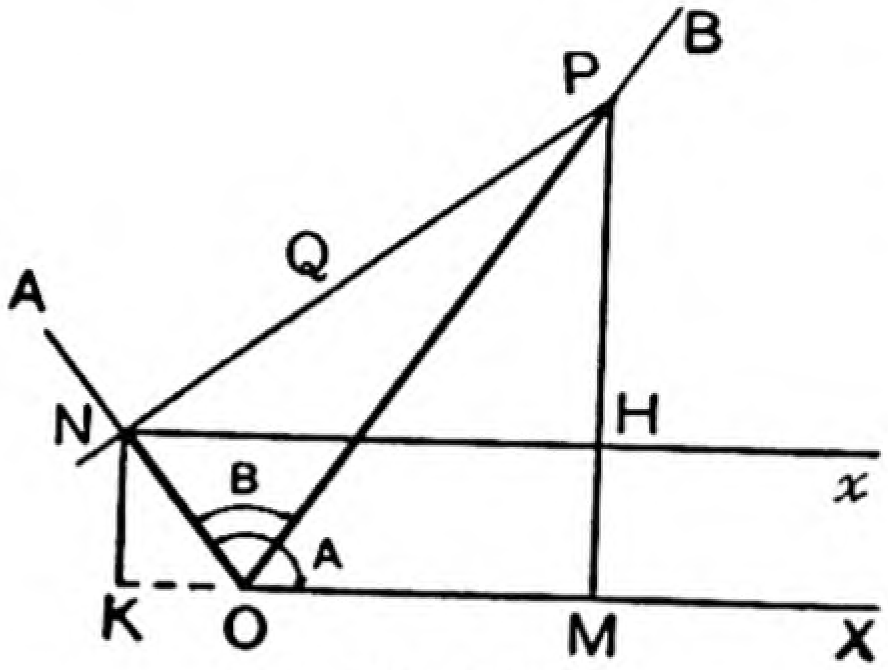


Fig. 98.

Take **P** on the bounding line **OB**, and draw perpendiculars as shown. Draw **N**x parallel to, and in the sense of, **OX**, and let **Q** be taken on **NP** or **NP** produced, so that $\angle$ **ANQ** is a right angle described *clockwise* from **NA**.

Then $\angle x\mathbf{NQ} = A-90°$.

Also the positive directions of **ON**, **NP** in defining the functions of B are indicated by **OA**, **NQ**, respectively, and we have

$$\sin(A-B) = \frac{\mathbf{MP}}{\mathbf{OP}} = \frac{\mathbf{KN}+\mathbf{HP}}{\mathbf{OP}} = \frac{\mathbf{KN}}{\mathbf{ON}}\cdot\frac{\mathbf{ON}}{\mathbf{OP}} + \frac{\mathbf{HP}}{\mathbf{NP}}\cdot\frac{\mathbf{NP}}{\mathbf{OP}}$$

$$= \sin A\cos B + \sin(A-90°)\sin B$$

$$= \sin A\cos B - \cos A\sin B.$$

$$\cos(A-B) = \frac{\mathbf{OM}}{\mathbf{OP}} = \frac{\mathbf{OK}+\mathbf{KM}}{\mathbf{OP}}\text{ (algebraically)} = \frac{\mathbf{OK}}{\mathbf{ON}}\cdot\frac{\mathbf{ON}}{\mathbf{OP}} + \frac{\mathbf{NH}}{\mathbf{NP}}\cdot\frac{\mathbf{NP}}{\mathbf{OP}}$$

$$= \cos A\cos B + \cos(A-90°)\sin B$$

$$= \cos A\cos B + \sin A\sin B.$$

$$\tan(A-B) = \frac{\mathbf{MP}}{\mathbf{OM}} = \frac{\mathbf{KN}+\mathbf{HP}}{\mathbf{OK}+\mathbf{KM}}$$

$$= \frac{\dfrac{\mathbf{KN}}{\mathbf{OK}} + \dfrac{\mathbf{HP}}{\mathbf{OK}}}{1+\dfrac{\mathbf{NH}}{\mathbf{HP}}\cdot\dfrac{\mathbf{HP}}{\mathbf{OK}}}.$$

Also $\dfrac{\mathbf{HP}}{\mathbf{NP}} = \sin(A-90°) = -\cos A = -\dfrac{\mathbf{OK}}{\mathbf{ON}}$, *algebraically*;

$$\therefore \frac{\mathbf{HP}}{\mathbf{OK}} = -\frac{\mathbf{NP}}{\mathbf{ON}} = -\tan B,\quad \frac{\mathbf{NH}}{\mathbf{HP}} = \cot(A-90°) = -\tan A,$$

$$\therefore \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}.$$

Illustrative Exercises.

Draw the figure and go through the proofs in the following cases :—

(1) When A is obtuse and B acute, and $A - B > 90°$.

(2) When A is acute and B obtuse, and $A - B$ a negative acute angle.

(3) When A and B are both obtuse, and $A > B$.

EXAMPLES XI.

1. Prove that $\sin(A+B) = \sin A \cos B + \cos A \sin B$, drawing the figure for the case in which A and B are each less than 90°, but $A+B$ is greater than 90°.

2. Prove, geometrically, that

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

each of the angles A and B being greater than 90° and less than 180°, and the angle $A+B$ less than 270°.

3. Prove, geometrically, that $\cos(A-B) = \cos A \cos B + \sin A \sin B$, A and B being angles in the second quadrant, *i.e.* the magnitude of each lying between 90° and 180°.

4. Prove that $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

5. Prove that $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, assuming the formula for $\sin(A-B)$ and $\cos(A-B)$.

6. Prove that $\tan 15° = 2 - \sqrt{3}$.

7. Assuming the formula for $\sin(A+B)$ and $\cos(A+B)$ in terms of the sines and cosines of A and B, deduce from them that for $\cot(A+B+C)$ in terms of the cotangents of A, B, and C.

8. Given that $\tan \alpha = a$, $\tan \beta = b$, $\tan \gamma = c$, find $\tan(\alpha+\beta+\gamma)$ in terms of a, b, and c.

9. If A, B, C are the angles of a triangle, and $2 \sin A \cos B = \sin C$, show that $A = B$.

10. If $\sin A = \frac{5}{13}$, $\sin B = \frac{3}{5}$, find $\cos(A+B)$.

11. If $\sin A = \frac{60}{61}$ and $\sin B = \frac{40}{41}$, find the sine and cosine of the sum and of the difference of A and B.

12. Prove that $\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \dfrac{\pi}{3}\right)$, and hence find the maximum value of $\cos\theta - \sqrt{3}\sin\theta$.

13. Prove that $\sin\theta + \cos\theta = \sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right)$.

14. Prove that $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$, where $\tan\alpha = \frac{a}{b}$; hence find the maximum value of the expression, and for what value of θ it occurs.

15. Find an expression for all the values of θ which satisfy the equation

$$\tan^2\theta - (\tan A + \cot A)\tan\theta + 1 = 0.$$

PROVE the following identities:—

16. $\sin(2A-B)\cos(2B-A)+\cos(2A-B)\sin(2B-A) = \sin(A+B)$.

17. $\sin(2A-B)\cos(2B-A)-\cos(2A-B)\sin(2B-A) = \sin 3(A-B)$.

18. $\cos(2A-B)\cos(2B-A)+\sin(2A-B)\sin(2B-A) = \cos 3(A-B)$.

19. $\dfrac{\sin(A+B)\cos(A-B)+\cos(A+B)\sin(A-B)}{\cos(A+B)\cos(A-B)-\sin(A+B)\sin(A-B)} = \tan 2A$.

20. $\dfrac{\cos(A+B)\sin(A-B)+\cos(A-B)\sin(A+B)}{\cos(A-B)\sin(A+B)-\cos(A+B)\sin(A-B)} = \dfrac{\sin A\cos A}{\sin B\cos B}$.

21. $\sin(n+1)B\sin(n-1)B+\cos(n+1)B\cos(n-1)B = \cos 2B$.

22. $\cos(135^\circ+A)+\sin(135^\circ-A) = 0$.

23. $\tan A - \tan\frac{A}{2} = \tan\frac{A}{2}\sec A$.

24. $\tan 2A\tan 3A\tan 5A = \tan 5A - \tan 3A - \tan 2A$.

25. $\tan 7A - \tan 4A - \tan 3A = \tan 7A\tan 4A\tan 3A$.

26. $\cot A - \cot 2A = \operatorname{cosec} 2A$.

27. $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$.

28. $\dfrac{1-\tan 2A\tan A}{1+\tan 2A\tan A} = \dfrac{\cos 3A}{\cos A}$.

29. $\dfrac{\sin A - \cos A\tan\frac{A}{2}}{\cos A + \sin A\tan\frac{A}{2}} = \tan\frac{A}{2}$.

30. $\dfrac{\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)} = \sin 2A$.

31. $\cos(15^\circ-\alpha)\sec 15^\circ - \sin(15^\circ-\alpha)\operatorname{cosec} 15^\circ = 4\sin\alpha$.

32. $(\cot A+\tan 2A)^2 = \cot^2 A(1+\tan^2 2A)$.

33. $1+\tan A\tan 2A = \tan 2A\cot A - 1 = \sec 2A$.

34. $\tan(A-B)+\tan(B-C)+\tan(C-A)$

$$= \tan(A-B)\tan(B-C)\tan(C-A).$$

35. $\sin(\alpha-\beta)\cos 2\beta+\cos(\alpha-\beta)\sin 2\beta$

$$= \sin(\beta-\alpha)\cos 2\alpha+\cos(\beta-\alpha)\sin 2\alpha.$$

36. $\dfrac{\sin(A-C)}{\cos A\cos C}+\dfrac{\sin(B-A)}{\cos B\cos A}+\dfrac{\sin(C-B)}{\cos C\cos B}=0.$

37. $\dfrac{\sin(A-C)}{\sin A\sin C}+\dfrac{\sin(B-A)}{\sin B\sin A}+\dfrac{\sin(C-B)}{\sin C\sin B}=0.$

38. $\sin A\,(\tan 2A\cot A+1)=\sin 3A\,(\tan 2A\cot A-1).$

39. $\sin 105^\circ+\cos 105^\circ=\cos 45^\circ.$

40. $\cos(A+B)\sin B-\cos(A+C)\sin C$
$$=\sin(A+B)\cos B-\sin(A+C)\cos C.$$

41. $\dfrac{\sin 2A}{\sin A}-\dfrac{\cos 2A}{\cos A}=\sec A.$

42. $\sin(\alpha+\beta)\cos\alpha-\cos(\alpha+\beta)\sin\alpha=\sin\beta.$

43. $\dfrac{\tan(\theta-\phi)+\tan\phi}{1-\tan(\theta-\phi)\tan\phi}=\tan\theta.$

44. $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A.$

45. $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A.$

46. $1-\tan^2 A\tan^2 B=\dfrac{\cos^2 B-\sin^2 A}{\cos^2 A\cos^2 B}.$

47. $\tan(A+B)\tan(A-B)=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}.$

48. $\sin A\sin 5A=\sin^2 3A-\sin^2 2A.$

49. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{\theta}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{\theta}{2}\right)=\dfrac{1}{\sqrt{2}}\sin\theta.$

CHAPTER XII.

MULTIPLE AND SUB-MULTIPLE ANGLES.

130. To express the trigonometric functions of an angle $2A$ in terms of those of A.

In the formulae for the sine, cosine, and tangent of $(A+B)$ make $B=A$: thus we obtain

$$\sin(A+A) = \sin A \cos A + \cos A \sin A;$$

$$\therefore \quad \sin 2A = 2 \sin A \cos A \quad \text{.......................(61)}$$

$$\cos(A+A) = \cos A \cos A - \sin A \sin A;$$

$$\therefore \quad \cos 2A = \cos^2 A - \sin^2 A \quad \text{.....................(62)}$$

or, remembering that $\sin^2 A + \cos^2 A = 1$,

$$\left.\begin{aligned} \cos 2A &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \end{aligned}\right\} \quad \text{.............. (63, 64)}$$

These four formulae are *very* important.

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A};$$

$$\therefore \quad \tan 2A = \frac{2\tan A}{1-\tan^2 A} \quad \text{.....................(65)}$$

Ex. To reduce to its simplest form

$$\frac{2\tan A}{1+\tan^2 A}.$$

Multiplying the numerator and denominator by $\cos^2 A$, we obtain

$$\frac{2\sin A\cos A}{\cos^2 A + \sin^2 A} = 2\sin A \cos A = \sin 2A.$$

131. **Geometrical proofs.** — When $A < 90°$, the above formulae may be very readily deduced from Fig. 99.

Let **OCA** be the diameter of a semicircle. Take $\angle \mathbf{AOP} = A$, and drop **PM** perpendicular on **OA**.

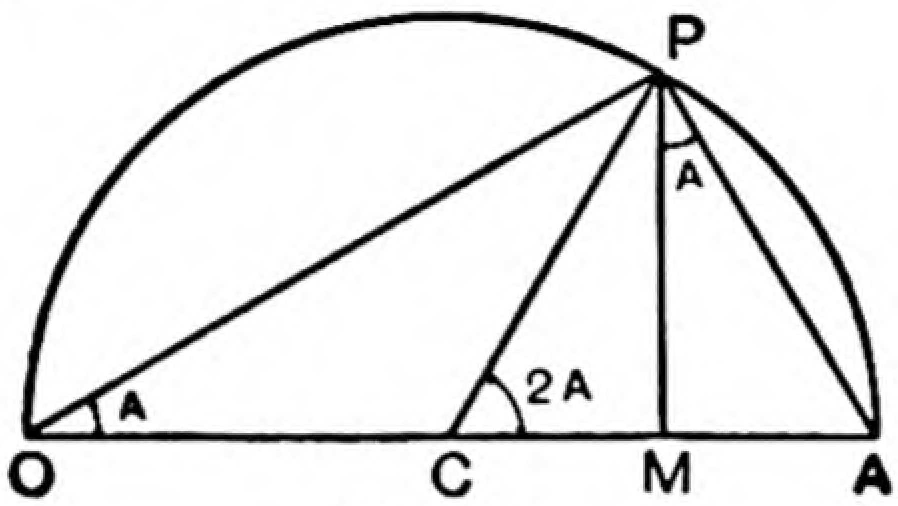


Fig. 99.

Then $$\angle \mathbf{ACP} = 2A;$$

$$\angle \mathbf{APO} = 90°;$$

and $$\angle \mathbf{APM} = 90° - \angle \mathbf{PAM} = \angle \mathbf{AOP} = A;$$

$$\sin 2A = \frac{\mathbf{MP}}{\mathbf{CP}} = \frac{\mathbf{MP}}{\mathbf{CA}} = 2 . \frac{\mathbf{MP}}{\mathbf{OA}} = 2 . \frac{\mathbf{MP}}{\mathbf{OP}} . \frac{\mathbf{OP}}{\mathbf{OA}}$$

$$= 2 \sin A \cos A.$$

$$\cos 2A = \frac{\mathbf{CM}}{\mathbf{CP}} = \frac{\mathbf{OM} - \mathbf{OC}}{\mathbf{OC}} = 2 . \frac{\mathbf{OM}}{\mathbf{OA}} - 1 = 2 . \frac{\mathbf{OM}}{\mathbf{OP}} . \frac{\mathbf{OP}}{\mathbf{OA}} - 1$$

$$= 2 \cos^2 A - 1;$$

$$\cos 2A = \frac{\mathbf{CM}}{\mathbf{CP}} = \frac{\mathbf{CA} - \mathbf{MA}}{\mathbf{CA}} = 1 - 2 . \frac{\mathbf{MA}}{\mathbf{OA}} = 1 - 2 . \frac{\mathbf{MA}}{\mathbf{PA}} . \frac{\mathbf{PA}}{\mathbf{OA}}$$

$$= 1 - 2 \sin^2 A;$$

$$\tan 2A = \frac{\mathbf{MP}}{\mathbf{CM}},$$

where $$\mathbf{CM} = \mathbf{CA} - \mathbf{MA} = \tfrac{1}{2}\mathbf{OA} - \mathbf{MA} = \tfrac{1}{2}(\mathbf{OM} + \mathbf{MA}) - \mathbf{MA}$$

$$= \tfrac{1}{2}(\mathbf{OM} - \mathbf{MA});$$

$$\therefore \quad \tan 2A = \frac{2 . \mathbf{MP}}{\mathbf{OM} - \mathbf{MA}} = \frac{2 . \dfrac{\mathbf{MP}}{\mathbf{OM}}}{1 - \dfrac{\mathbf{MA}}{\mathbf{PM}} \cdot \dfrac{\mathbf{PM}}{\mathbf{OM}}}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}.$$

132. To express the sine, cosine, and tangent of $\frac{1}{2}A$ in terms of cos A.

In the first place, starting with the "$2A$ formulae" of § 130, we notice that the following identities—

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \quad\text{.......................(i)}$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \quad\text{...................(ii)}$$

$$= 2\cos^2\frac{A}{2} - 1 \quad\text{.......................(iii)}$$

$$= 1 - 2\sin^2\frac{A}{2} \quad\text{.......................(iv)}$$

are other forms of (61–64) got by writing $\frac{A}{2}$ for A.

From (iv), $\quad 2\sin^2\frac{1}{2}A = 1-\cos A,$

or $$\sin^2\frac{A}{2} = \frac{1-\cos A}{2} \quad\text{..................(66)}$$

From (iii), $\quad 2\cos^2\frac{1}{2}A = 1+\cos A,$

or $$\cos^2\frac{A}{2} = \frac{1+\cos A}{2} \quad\text{..................(67)}$$

By division, $$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} \quad\text{..................(68)}$$

Extracting the square roots of (66, 67, 68), we have*

$$\sin\frac{A}{2} = \pm\sqrt{\left\{\frac{1-\cos A}{2}\right\}} \quad\text{.............. (66A)}$$

$$\cos\frac{A}{2} = \pm\sqrt{\left\{\frac{1+\cos A}{2}\right\}} \quad\text{.............. (67A)}$$

$$\tan\frac{A}{2} = \pm\sqrt{\left\{\frac{1-\cos A}{1+\cos A}\right\}} \quad\text{.............. (68A)}$$

Illustrative Exercise.

Prove (66)–(68) geometrically by taking $\angle$ AOP $= \frac{1}{2}A$ in Fig. 99.

---

* We recommend that equations involving radicals, such as these, be remembered in their squared forms unless the other forms are preferred.

133. If we know the value of A, we know the signs of $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$, etc., as will be evident from the following examples:—

Ex. 1. To find the sine and cosine of $22\frac{1}{2}°$.

Since $22\frac{1}{2}°$ is in the first quadrant, all its functions are *positive*. Putting $A = 45°$ in the above, we have, therefore,

$$\sin 22\tfrac{1}{2}° = +\sqrt{\left(\frac{1-\cos 45°}{2}\right)} = +\sqrt{\left(\frac{1}{2}-\frac{1}{2\sqrt{2}}\right)} = \sqrt{\left(\frac{2-\sqrt{2}}{4}\right)}$$
$$= +\tfrac{1}{2}\sqrt{(2-\sqrt{2})}.$$

Similarly, $$\cos 22\tfrac{1}{2}° = +\tfrac{1}{2}\sqrt{(2+\sqrt{2})}.$$

Ex. 2. To find the sine and cosine of $157\frac{1}{2}°$.

Since $157\frac{1}{2}°$ is in the second quadrant, its sine is positive, and its cosine negative. Putting $\frac{1}{2}A = 157\frac{1}{2}°$, we have

$$A = 315° = 360°-45°;$$

hence $\cos A = \cos 45°$, and the expressions are numerically the same as in Ex. 1, but they give

$$\sin 157\tfrac{1}{2}° = +\tfrac{1}{2}\sqrt{(2-\sqrt{2})}, \quad \cos 157\tfrac{1}{2}° = -\tfrac{1}{2}\sqrt{(2+\sqrt{2})}.$$

These results might have been deduced from Ex. 1 by the relation

$$157\tfrac{1}{2}° = 180°-22\tfrac{1}{2}°.$$

Ex. 3. To find the sine and cosine of $292\frac{1}{2}°$.

The given angle being in the fourth quadrant, its sine is negative and cosine positive.

Also, $$\cos 585° = \cos(3\times 180°+45°) = -\cos 45°.$$

Therefore

$$\sin 292\tfrac{1}{2}° = -\sqrt{\left(\frac{1+\cos 45°}{2}\right)} = -\sqrt{\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)} = -\frac{\sqrt{(2+\sqrt{2})}}{2},$$

$$\cos 292\tfrac{1}{2}° = \sqrt{\left(\frac{1-\cos 45°}{2}\right)} = \sqrt{\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)} = \frac{\sqrt{(2-\sqrt{2})}}{2}.$$

134. To prove that

$$\tan\frac{A}{2} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A} \quad \text{.........(69, 70)}$$

$$\frac{\sin A}{1+\cos A} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}} = \tan\frac{A}{2} \quad \text{...........(69)}$$

$$\frac{1-\cos A}{\sin A} = \frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \tan\frac{A}{2} \quad \text{...........(70)}$$

[*Geometrically thus:* In Fig. 99 (p. 143), let $\angle \mathsf{ACP} = A$, and therefore $\angle \mathsf{AOP} = \frac{1}{2}A$. Then $\angle \mathsf{MPA} = \frac{1}{2}A$, and

$$\tan\frac{A}{2} = \frac{\mathsf{MP}}{\mathsf{OM}} = \frac{\mathsf{MP}}{\mathsf{OC}+\mathsf{CM}} = \frac{\dfrac{\mathsf{MP}}{\mathsf{CP}}}{1+\dfrac{\mathsf{CM}}{\mathsf{CP}}} = \frac{\sin A}{1+\cos A},$$

$$\tan\frac{A}{2} = \frac{\mathsf{MA}}{\mathsf{MP}} = \frac{\mathsf{CA}-\mathsf{CM}}{\mathsf{MP}} = \frac{1-\dfrac{\mathsf{CM}}{\mathsf{CP}}}{\dfrac{\mathsf{MP}}{\mathsf{CP}}} = \frac{1-\cos A}{\sin A}.\Big]$$

Ex. To find $\tan 22\frac{1}{2}^\circ$ and $\cot 22\frac{1}{2}^\circ$.
By (70),

$$\tan 22\tfrac{1}{2}^\circ = \frac{1-\cos 45^\circ}{\sin 45^\circ} = \frac{1-1/\sqrt{2}}{1/\sqrt{2}} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1.$$

By (69),

$$\cot 22\tfrac{1}{2}^\circ = \frac{1}{\tan 22\frac{1}{2}^\circ} = \frac{1+\cos 45^\circ}{\sin 45^\circ} = \frac{1+1/\sqrt{2}}{1/\sqrt{2}} = \frac{\sqrt{2}+1}{1} = \sqrt{2}+1.$$

135. To express the trigonometric functions of $3A$ in terms of the corresponding functions of A.

$$\begin{aligned}\sin 3A &= \sin(2A+A) = \sin 2A\cos A+\cos 2A\sin A\\ &= 2\sin A\cos^2 A+\sin A\,(1-2\sin^2 A)\\ &= 2\sin A\,(1-\sin^2 A)+\sin A\,(1-2\sin^2 A).\end{aligned}$$

$$\therefore\quad \mathbf{\sin 3A = 3\sin A-4\sin^3 A} \quad\ldots\ldots\ldots\ldots\mathbf{(71)}$$

$$\begin{aligned}\cos 3A &= \cos 2A\cos A-\sin 2A\sin A\\ &= (2\cos^2 A-1)\cos A-2\sin A\cos A\sin A\\ &= (2\cos^2 A-1)\cos A-2\cos A\,(1-\cos^2 A)\\ &\qquad (\because\quad \sin^2 A = 1-\cos^2 A).\end{aligned}$$

$$\therefore\quad \mathbf{\cos 3A = 4\cos^3 A-3\cos A} \quad\ldots\ldots\ldots\ldots\mathbf{(72)}$$

These two formulae should be remembered.

In like manner we may obtain

$$\mathbf{\tan 3A = \frac{3\tan A-\tan^3 A}{1-3\tan^2 A}} \quad\ldots\ldots\ldots\ldots\mathbf{(73)}$$

The proof is similar to that of (71, 72), but the result is less important.

136. To determine the sine and cosine of $\frac{1}{2}A$, having given $\sin A$.

We shall now prove that

$$\cos\frac{A}{2}+\sin\frac{A}{2}=\pm\sqrt{(1+\sin A)} \quad \text{..........(74)}$$

$$\cos\frac{A}{2}-\sin\frac{A}{2}=\pm\sqrt{(1-\sin A)} \quad \text{..........(75)}$$

Proof.—For

$$\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2=\cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}$$
$$=1+\sin A, \qquad \text{by (61)}$$

and, similarly, $$\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2=1-\sin A.$$

*137. The signs to be given to the radicals depend not only on the signs of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$, but on *which of them is the greater*. (Cf. Exs. 1, 2, below.) When the right signs are assigned, we have, by addition and subtraction, respectively,

$$\cos\frac{A}{2}=\tfrac{1}{2}\{\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\},$$

$$\sin\frac{A}{2}=\tfrac{1}{2}\{\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}\}.$$

But the equations should *always* be used in forms (74, 75), or, better still, in their squared forms.

Ex. 1. To find $\sin 15°$, having given $\sin 30° = \frac{1}{2}$.
Here the formula gives

$$\cos 15°+\sin 15° = \pm\sqrt{\tfrac{3}{2}},$$
$$\cos 15°-\sin 15° = \pm\sqrt{\tfrac{1}{2}}.$$

Now $15°$ is in the first quadrant; hence $\sin 15°$ and $\cos 15°$ are positive, and so also is their sum.

To decide the sign of $\cos 15°-\sin 15°$, we notice that it is positive or negative according as

$$\cos 15° > \text{ or } < \sin 15°;$$

i.e. as $$1 > \text{ or } < \tan 15°;$$

i.e. as $$\tan^{-1} 1 \text{ or } 45° > \text{ or } < 15°.$$

Since $45° > 15°$, the positive sign must be taken; hence

$$\cos 15°+\sin 15° = +\sqrt{\tfrac{3}{2}},$$
$$\cos 15°-\sin 15° = +\sqrt{\tfrac{1}{2}}.$$

Solving these equations, we have

$$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}, \quad \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

as in § 119.

Ex. 2. To find sin 75° and cos 75°.

Here, if $\frac{1}{2}A = 75^\circ$, then $A = 150^\circ = 180^\circ - 30^\circ$.

Hence, $\sin A = \sin 30^\circ = \frac{1}{2}$, and the formulae are identical with those of Ex. 1, viz.

$$\cos 75^\circ + \sin 75^\circ = \pm\sqrt{\tfrac{3}{2}},$$
$$\cos 75^\circ - \sin 75^\circ = \pm\sqrt{\tfrac{1}{2}}.$$

Now, sin 75° and cos 75° are both positive, but, reasoning as in Ex. 1, we easily see that $\cos 75^\circ < \sin 75^\circ$.

Hence the second radical must be taken negative, that is,

$$\cos 75^\circ + \sin 75^\circ = +\sqrt{\tfrac{3}{2}},$$
$$\cos 75^\circ - \sin 75^\circ = -\sqrt{\tfrac{1}{2}},$$

giving this time

$$\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \quad \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

as in § 117.

NOTE.—The two examples above illustrate why it is that the ambiguities of sign arise in equations (74, 75). Thus, if we are only given that $\sin A = \frac{1}{2}$, A may have any of the following values [obtained from the formula $n.180^\circ + (-1)^n.30^\circ$], viz.

$A = 30^\circ,\ 180^\circ - 30^\circ,\ 360^\circ + 30^\circ,\ 540^\circ - 30^\circ,\ 720^\circ + 30^\circ,\ 900^\circ - 30^\circ$, etc.,

corresponding to which

$\frac{1}{2}A = 15^\circ,\ 90^\circ - 15^\circ,\ 180^\circ + 15^\circ,\ 270^\circ - 15^\circ,\ 360^\circ + 15^\circ,\ 450^\circ - 15^\circ$, etc.

In this series, the first four angles have different sines and cosines, but all the other angles are coterminal with them. It will be interesting to verify as an exercise that

$$\left.\begin{array}{l}\cos 15^\circ + \sin 15^\circ = +\sqrt{(1+\frac{1}{2})} \\ \cos 15^\circ - \sin 15^\circ = +\sqrt{(1-\frac{1}{2})}\end{array}\right\}, \quad \left.\begin{array}{l}\cos 75^\circ + \sin 75^\circ = +\sqrt{(1+\frac{1}{2})} \\ \cos 75^\circ - \sin 75^\circ = -\sqrt{(1-\frac{1}{2})}\end{array}\right\},$$

$$\left.\begin{array}{l}\cos 195^\circ + \sin 195^\circ = -\sqrt{(1+\frac{1}{2})} \\ \cos 195^\circ - \sin 195^\circ = -\sqrt{(1-\frac{1}{2})}\end{array}\right\}, \quad \left.\begin{array}{l}\cos 255^\circ + \sin 255^\circ = -\sqrt{(1+\frac{1}{2})} \\ \cos 255^\circ - \sin 255^\circ = +\sqrt{(1-\frac{1}{2})}\end{array}\right\},$$

***138. To decide the signs of the expressions**

$$\cos \tfrac{1}{2}A + \sin \tfrac{1}{2}A \quad \text{and} \quad \cos \tfrac{1}{2}A - \sin \tfrac{1}{2}A,$$

for any given value of A, we may use the identities

$$\sin(45^\circ + \tfrac{1}{2}A) = \sin 45^\circ \cos \tfrac{1}{2}A + \cos 45^\circ \sin \tfrac{1}{2}A$$
$$= \frac{1}{\sqrt{2}}\cos \tfrac{1}{2}A + \frac{1}{\sqrt{2}}\sin \tfrac{1}{2}A = \frac{\cos \frac{1}{2}A + \sin \frac{1}{2}A}{\sqrt{2}},$$

$$\cos(45^\circ+\tfrac{1}{2}A) = \cos 45^\circ \cos \tfrac{1}{2}A - \sin 45^\circ \sin \tfrac{1}{2}A$$
$$= \frac{1}{\sqrt{2}} \cos \tfrac{1}{2}A - \frac{1}{\sqrt{2}} \sin \tfrac{1}{2}A = \frac{\cos \frac{1}{2}A - \sin \frac{1}{2}A}{\sqrt{2}}.$$

In these expressions $\sqrt{2}$ is taken with the positive sign because it is introduced through the sine and cosine of 45°, which are positive. Hence the expressions

$$\cos \tfrac{1}{2}A + \sin \tfrac{1}{2}A \quad \text{and} \quad \cos \tfrac{1}{2}A - \sin \tfrac{1}{2}A$$

have the same signs as $\sin(45^\circ+\frac{1}{2}A)$ and $\cos(45^\circ+\frac{1}{2}A)$, respectively.

Ex. Taking the angles 195° and 255° of the note of the preceding article, we have

$\cos 195^\circ + \sin 195^\circ = \sqrt{2} \sin(45^\circ+195^\circ) = \sqrt{2} \sin 240^\circ$, and is *negative*,
$\cos 195^\circ - \sin 195^\circ = \sqrt{2} \cos(45^\circ+195^\circ) = \sqrt{2} \cos 240^\circ$, and is *negative*,
$\cos 255^\circ + \sin 255^\circ = \sqrt{2} \sin(45^\circ+255^\circ) = \sqrt{2} \sin 300^\circ$, and is *negative*,
$\cos 255^\circ - \sin 255^\circ = \sqrt{2} \cos(45^\circ+255^\circ) = \sqrt{2} \cos 300^\circ$, and is *positive*.

EXAMPLES XII.

1. Prove that $\tan^2 \dfrac{A}{2} = \dfrac{1-\cos A}{1+\cos A}$.

2. Solve the equation $\cos 2x = \cos^2 x$.

3. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, find the values of $\tan(2A+B)$ and $\tan(2A-B)$.

4. Prove the formula $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

5. Show, geometrically, that $\sin 2A$ is $< 2 \sin A$.

6. If $\cos 4\theta = n$, find from this a formula for $\sin\theta$, and apply it to find $\sin\dfrac{\pi}{8}$ and $\sin\dfrac{5\pi}{8}$.

7. If $\tan 2\theta = n$, find $\tan\theta$ in terms of n. If $n = \sqrt{3}$ and is positive, find from the former equation all the values of θ between 0 and 360°, and show that the same values will be obtained from the second equation.

8. Show that $8(\cos^8 A - \sin^8 A) = \cos 6A + 7\cos 2A$.

9. Prove the formula $\sin A = 3\sin\dfrac{A}{3} - 4\sin^3\dfrac{A}{3}$.

10. If the value of $\cos A$ be given, show geometrically that there are in general three different values of $\cos\dfrac{A}{3}$, and point out all the angles to which they belong.

11. Show, geometrically, that if the value of $\tan A$ be given, nothing else being known about the angle A, there are four values of $\sin\frac{A}{2}$, and explain why the sum of the squares of these values is 2.

12. If $20 \tan A = 21$, find all the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$.

13. If $\tan \alpha = \frac{1}{5}$, $\tan \beta = \frac{1}{239}$, show that $\tan(4\alpha - \beta) = 1$.

14. Replace the ambiguous signs in the formulae of § 136 by the proper signs when A is between 270° and 360°.

Prove the following identities (15–90):—

15. $\dfrac{\sin 2A}{1+\cos 2A} = \tan A.$

16. $\dfrac{\sin 2A}{1-\cos 2A} = \cot A.$

17. $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}.$

18. $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}.$

19. $(\cos A+\cos B)(\cos A-\cos B) = \frac{1}{2}(\cos 2A - \cos 2B).$

20. $(\cos A+\cos B)(\cos 2A+\cos 2B)(\cos 2^2A+\cos 2^2B)(\cos 2^3A+\cos 2^3B)$
$$\ldots(\cos 2^{n-1}A+\cos 2^{n-1}B) = \frac{\cos 2^n A-\cos 2^n B}{2^n(\cos A-\cos B)}.$$

21. $\sin A = \dfrac{\sin 2A\cos A}{1+\cos 2A}.$

22. $\dfrac{(\operatorname{cosec} A+\sec A)^2}{\operatorname{cosec}^2 A+\sec^2 A} = 1+\sin 2A.$

23. $\tan A = \dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A}.$

24. $1+\tan 2A\tan A = \sec 2A.$

25. $\operatorname{cosec} 2A - \cot 2A = \tan A.$

26. $\cot A - \tan A = 2\cot 2A.$

27. $\tan A = (1+\sec A)\tan\dfrac{A}{2}.$

28. $\cot A+\operatorname{cosec} A = \cot\dfrac{A}{2}.$

29. $\sin^2 2A = 2\cos^2 A(1-\cos 2A).$

30. $\dfrac{1+\sin A-\cos A}{1+\sin A+\cos A} = \tan\dfrac{A}{2}.$

31. $\dfrac{\tan 5A+\tan 3A}{\tan 5A-\tan 3A} = 4\cos 2A\cos 4A.$

32. $\sec\left(\dfrac{\pi}{4}+A\right)\sec\left(\dfrac{\pi}{4}-A\right) = 2\sec 2A.$

33. $\dfrac{\sin 8A}{2\sin A} = \cos A+\cos 3A+\cos 5A+\cos 7A.$

34. $2-2\tan\theta\cot 2\theta = \sec^2\theta.$

35. $\cos^2\dfrac{A}{2}\left(1+\tan\dfrac{A}{2}\right)^2 = 1+\sin A.$

36. $1+\cos 4A = 2\cos 2A(1-2\sin^2 A).$

37. $\dfrac{\tan 3A}{\tan A} = \dfrac{2\cos 2A+1}{2\cos 2A-1}.$

38. $2\tan A+\tan\dfrac{A}{2} = \cot\dfrac{A}{2} - 4\cot 2A.$

39. $\dfrac{\sin 2A}{1+\cos 2A}\,\dfrac{\cos A}{1+\cos A} = \tan\dfrac{A}{2}$.

40. $2\cot A - 2\cot 2A = \sec A \operatorname{cosec} A$.

41. $\dfrac{\tan\left(\dfrac{\pi}{4}+A\right)}{\tan\left(\dfrac{\pi}{4}-A\right)} = \dfrac{2\cos A+\sin A+\sin 3A}{2\cos A-\sin A-\sin 3A}$.

42. $(1+\cot A+\operatorname{cosec} A)\,(1+\cot A-\operatorname{cosec} A) = \cot\dfrac{A}{2} - \tan\dfrac{A}{2}$.

43. $\dfrac{\sin A}{\sin\dfrac{A}{8}} = 2^3\cos\dfrac{A}{2}\cos\dfrac{A}{4}\cos\dfrac{A}{8}$.

44. $\sin^2\alpha-\sin^2\beta = \sin\alpha\sin(2\beta+\alpha)-\sin\beta\sin(2\alpha+\beta)$.

45. $8(\sin^4 A-\sin^2 A)+1 = \cos 4A$.

46. $4\sin 20^\circ\sin 40^\circ\sin 80^\circ = \sin 60^\circ$.

47. $\sec 2A-\frac{1}{2}\tan 2A\sin 2A = \dfrac{\cot^2 A+\tan^2 A}{\cot^2 A-\tan^2 A}$.

48. $\cos 2A = \dfrac{1-6\tan^2\dfrac{A}{2}+\tan^4\dfrac{A}{2}}{\left(1+\tan^2\dfrac{A}{2}\right)^2}$.

49. $\tan 3A = \dfrac{3\tan A\sec^2 A-4\tan^3 A}{4-3\sec^2 A}$.

50. $\dfrac{\tan A+\sec A}{\cot A+\operatorname{cosec} A} = \tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)\tan\dfrac{A}{2}$.

51. $\dfrac{\cos\left(\dfrac{\pi}{4}+A\right)}{\cos\left(\dfrac{\pi}{4}-A\right)} = \sec 2A-\tan 2A$.

52. $\tan\left(\dfrac{\pi}{4}-\dfrac{A}{2}\right)+\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right) = 2\sec A$.

53. $\cot(A+15^\circ)-\tan(A-15^\circ) = \dfrac{4\cos 2A}{2\sin 2A+1}$.

54. $\tan 3\alpha = \tan\left(\dfrac{\pi}{3}-\alpha\right)\tan\alpha\tan\left(\dfrac{\pi}{3}+\alpha\right)$.

55. $\sin 3A = 4\sin A\sin(60^\circ+A)\sin(60^\circ-A)$.

56. $\cos 3A = 4\cos A\sin(30^\circ-A)\sin(30^\circ+A)$.

57. $\tan(60^\circ+A)\tan(60^\circ-A)=\dfrac{2\cos 2A+1}{2\cos 2A-1}$.

58. $\cos 4A = 8\cos^4 A-8\cos^2 A+1$.

59. $2(\operatorname{cosec} 4A+\cot 4A)=\cot A-\tan A$.

60. $\cos 4A=\cos^4 A+\sin^4 A-6\sin^2 A\cos^2 A$.

61. $\cos^2 A-\cos A\cos(60^\circ+A)+\sin^2(30^\circ-A)=\frac{3}{4}$.

62. $\cos 2A\cos 2B=\cos 2A+\cos 2B+4\sin^2 A\sin^2 B-1$.

63. $\dfrac{\cos\frac{A}{2}-\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}}=\sec A-\tan A$.

64. $\sin^2 A\cos 2B+2\sin A\sin B\cos(B-A)+\sin^2 B\cos 2A=\sin^2(A+B)$.

65. $(2\cos A+1)(2\cos A-1)(2\cos 2A-1)=2\cos 4A+1$.

66. $\sec^2\dfrac{\pi-A}{4}+\sec^2\dfrac{\pi+A}{4}=\sec^2\dfrac{\pi-A}{4}\sec^2\dfrac{\pi+A}{4}$.

67. $\sec A+\sec(120^\circ+A)+\sec(120^\circ-A)+3\sec 3A=0$.

68. $\tan 2A=(\sec 2A+1)\sqrt{\sec^2 A-1}$.

69. $\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)=\left(\dfrac{1+\sin A}{1-\sin A}\right)^{\frac{1}{2}}$.

70. $\cos^3 2A+3\cos 2A=4(\cos^6 A-\sin^6 A)$.

71. $5+3\cos 4A=8(\cos^6 A+\sin^6 A)$.

72. $\cos^2 A\cos^2 B+\sin^2 A\sin^2 B=\frac{1}{2}(1+\cos 2A\cos 2B)$.

73. $\sin^3 A\dfrac{\cos 3A}{3}+\cos^3 A\dfrac{\sin 3A}{3}=\dfrac{\sin 4A}{4}$.

74. $\sin^3 A+\sin^3(120+A)+\sin^3(240+A)=-\frac{3}{4}\sin 3A$

75. $\tan\theta=\tan\dfrac{\theta}{3}\cot\dfrac{1}{3}\left(\dfrac{\pi}{2}-\theta\right)\cot\dfrac{1}{3}\left(\dfrac{\pi}{2}+\theta\right)$.

76. $\{\sec\theta+\operatorname{cosec}\theta(1+\sec\theta)\}\left(1-\tan^2\dfrac{\theta}{2}\right)\left(1-\tan^2\dfrac{\theta}{4}\right)$
$$=\left(\sec\frac{\theta}{2}+\operatorname{cosec}\frac{\theta}{2}\right)\sec^2\frac{\theta}{4}.$$

77. $\sec^2\dfrac{\theta}{2}\sec\theta\left(\cot^2\dfrac{\theta}{2}-\cot^2\dfrac{3\theta}{2}\right)=8\left(1+\cot^2\dfrac{3\theta}{2}\right)$.

78. $\dfrac{\cos A\cot A-\sin A\tan A}{\cos A\cot A+\sin A\tan A}\times\dfrac{2-\sin 2A}{2+\sin 2A}=\tan\left(\dfrac{\pi}{4}-A\right)$.

79. $\cos A-\tan\dfrac{A}{2}\sin A=\cos 2A+\tan\dfrac{A}{2}\sin 2A$.

80. $$\frac{(\sec A \sec B+\tan A \tan B)^2-(\tan A \sec B+\sec A \tan B)^2}{2\,(1+\tan^2 A \tan^2 B)-\sec^2 A \sec^2 B} = \frac{\sec 2A \sec 2B}{\sec^2 A \sec^2 B}.$$

81. $4\,(\cos^3 10^\circ+\sin^3 20^\circ) = 3\,(\cos 10^\circ+\sin 20^\circ).$

82. $(1-\tan^2 A)\,(\tan 2A-2\tan A) = 2\tan^3 A.$

83. $\cot^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)=\dfrac{2\operatorname{cosec} 2\theta-\sec\theta}{2\operatorname{cosec} 2\theta+\sec\theta}.$

84. $1\pm\sin\theta = 2\sin^2\left(\dfrac{\pi}{4}\pm\dfrac{\theta}{2}\right).$

85. $(2\cos A-1)\,(2\cos 2A-1)\,(2\cos 2^2 A-1)\ldots n$ factors
$$=\frac{2\cos 2^n A+1}{2\cos A+1}.$$

86. $\sqrt{\dfrac{a-b}{a+b}}+\sqrt{\dfrac{a+b}{a-b}}=\dfrac{2\cos A}{\sqrt{\cos 2A}},$ if $a\sin A = b\cos A.$

87. $\dfrac{1}{2}\tan\dfrac{\theta}{2}+\dfrac{1}{4}\tan\dfrac{\theta}{4}=\dfrac{1}{4}\cot\dfrac{\theta}{4}-\cot\theta.$

88. $\cos^2 A+\sin^2 A\cos 2B = \cos^2 B+\sin^2 B\cos 2A.$

89. $\sin^2 A-\cos^2 A\cos 2B = \sin^2 B-\cos^2 B\cos 2A.$

90. $\sin 3A-\cos 3A+3\,(\cos A+\sin A) = 2\,(\sin A+\cos A)^3.$

CHAPTER XIII.

SUMS AND PRODUCTS OF TWO SINES OR COSINES.

139. Starting with the four formulae of Chap. XI.,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad\text{..........(47)}$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad\text{..........(48)}$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad\text{..........(49)}$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad\text{..........(50)}$$

we shall now deduce certain very important formulae whereby expressions involving the sum, difference, or product of two sines or two cosines, or the product of a cosine and a sine, can often be considerably simplified.

140. Expressing products of trigonometric functions as sums.—Taking the first and second of these four formulae, and first adding them and then subtracting, and proceeding similarly with the third and fourth formulae, we obtain the four relations—

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \quad\text{......(76)}$$
$$\sin(A+B) - \sin(A-B) = 2 \cos A \sin B \quad\text{......(77)}$$
$$\cos(A-B) + \cos(A+B) = 2 \cos A \cos B \quad\text{......(78)}$$
$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B \quad\text{......(79)}$$

Writing (76–79) backward, we have

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B) \quad\text{......(76A)}$$
$$2 \cos A \sin B = \sin(A+B) - \sin(A-B) \quad\text{......(77A)}$$
$$2 \cos A \cos B = \cos(A-B) + \cos(A+B) \quad\text{......(78A)}$$
$$2 \sin A \sin B = \cos(A-B) - \cos(A+B) \quad\text{......(79A)}$$

Caution.—*Note the order* of the terms on the right-hand side of equations 78A and 79A. In the first quadrant, *the greater the angle the less the cosine;* hence *the cosine of the smaller angle is written first* to get a positive result.

These formulae are often remembered in words:—

twice sine × cosine = sin sum + sin diff. ......(76*a*)

twice cosine × sine = sin sum − sin diff. ......(77*a*)

twice product of cosines = cos diff. + cos sum ......(78*a*)

twice product of sines = cos diff. − cos sum ......(79*a*)

[*Notice again that cos diff. takes precedence of cos sum.*]

Ex. 1. Express $2 \sin 40^\circ \cos 60^\circ$ as the difference of two sines.

This raises the question when to use (76*a*), and when (77*a*). The safest rule is to *write the larger angle first.* Thus,

$$2 \sin 40^\circ \cos 60^\circ = 2 \cos 60^\circ \sin 40^\circ = \sin 100^\circ - \sin 20^\circ \ldots \text{(by 77}a\text{)}.$$

Even without this precaution, no mistake can be made if due regard be paid to sign. Thus (by 76*a*),

$$2 \sin 40^\circ \cos 60^\circ = \sin(40^\circ + 60^\circ) + \sin(40^\circ - 60^\circ) = \sin 100^\circ + \sin(-20^\circ)$$
$$= \sin 100^\circ - \sin 20^\circ, \text{ as before.}$$

Ex. 2. $$\sin\left(A + \frac{B}{2}\right) \sin\left(C + \frac{B}{2}\right)$$
$$= \tfrac{1}{2}\{\cos(A-C) - \cos(A+B+C)\} \ldots \text{(by 79}a\text{)}.$$

Ex. 3. Express $\sin A \sin B \sin C$ and $\cos A \cos B \cos C$ in forms not involving products.

By (79*a*),

$$\sin A \sin B \sin C = \tfrac{1}{2} \sin A\{\cos(B-C) - \cos(B+C)\}$$
$$= \tfrac{1}{4}\{\sin(A+B-C) + \sin(A-B+C) - \sin(A+B+C) - \sin(A-B-C)\};$$

by (76*a*), or, as it may be written more symmetrically,

$$= \tfrac{1}{4}\{\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) - \sin(A+B+C)\}.$$

Again, $$\cos A \cos B \cos C = \tfrac{1}{2} \cos A\{\cos(B-C) + \cos(B+C)\}$$
$$= \tfrac{1}{4}\{\cos(B+C-A) + \cos(C+A-B) + \cos(A+B-C) + \cos(A+B+C)\}.$$

[These results may be written in a shorter form by introducing the letter S to represent the semi-sum of A, B, C, *i.e.* $\tfrac{1}{2}(A+B+C)$. Thus $A+B+C = 2S$, $B+C-A = 2(S-A)$, and so on, and the identities become

$$\sin A \sin B \sin C$$
$$= \tfrac{1}{4}\{\sin 2(S-A) + \sin 2(S-B) + \sin 2(S-C) - \sin 2S\},$$
$$\cos A \cos B \cos C$$
$$= \tfrac{1}{4}\{\cos 2(S-A) + \cos 2(S-B) + \cos 2(S-C) + \cos 2S\}.]$$

141. Sum and difference of two sines or cosines.—In the last article we expressed the product of two trigonometric functions (sines or cosines) as a sum or difference. We shall now, conversely, express the sum or difference of two sines or two cosines as a product.

In formulae (76–79) put

$$S = A + B, \quad T = A - B,$$

$$\therefore \quad A = \frac{S+T}{2}, \quad B = \frac{S-T}{2};$$

hence we may rewrite the formulae thus:—

$$\sin S + \sin T = 2 \sin \frac{S+T}{2} \cos \frac{S-T}{2} \quad \text{..........(80)}$$

$$\sin S - \sin T = 2 \cos \frac{S+T}{2} \sin \frac{S-T}{2} \quad \text{..........(81)}$$

$$\cos T + \cos S = 2 \cos \frac{S+T}{2} \cos \frac{S-T}{2} \quad \text{..........(82)}$$

$$\cos T - \cos S = 2 \sin \frac{S+T}{2} \sin \frac{S-T}{2} \quad \text{..........(83)}$$

Caution.—*Note carefully the order of the cosines in the two last formulae.*

These formulae are of fundamental importance, and require a seemingly exorbitant amount of practice before one is able to use them with the necessary facility.

These formulae should also be known in words:—

Sum of sines = 2 sin semi-sum cos semi-difference ....(80)

Diff. ,, = 2 cos semi-sum sin semi-difference ....(81)

Sum of cosines = 2 cos semi-sum cos semi-difference ....(82)

Diff. ,, or cos 2nd—cos 1st = 2 sin semi-sum sin semi-difference ....(83)

The latter rendering of (83) also draws attention to the peculiar order of the cosines. Even if the student repeats these formulae by putting them into words, *e.g.* thus—

cosine of 2nd angle − cosine of 1st angle

= twice product of sines of half (1st+2nd) and half (1st−2nd),
he will not get too much practice in them.

Ex. 1. Express $1+\sin A$ and $1-\sin A$ as the products of a cosine and sine.

$$1+\sin A = \sin 90^\circ+\sin A = 2\sin\tfrac{1}{2}(90^\circ+A)\cos\tfrac{1}{2}(90^\circ-A)$$
$$= 2\sin(45^\circ+\tfrac{1}{2}A)\cos(45^\circ-\tfrac{1}{2}A),$$
$$1-\sin A = \sin 90^\circ-\sin A = 2\cos\tfrac{1}{2}(90^\circ+A)\sin\tfrac{1}{2}(90^\circ-A)$$
$$= 2\cos(45^\circ+\tfrac{1}{2}A)\sin(45^\circ-\tfrac{1}{2}A).$$

Ex. 2. Simplify $\dfrac{\cos A-\cos(A+2B)}{\sin A+\sin(A+2B)}$.

The expression

$$= \frac{2\sin\frac{1}{2}(A+2B+A)\sin\frac{1}{2}(A+2B-A)}{2\sin\frac{1}{2}(A+2B+A)\cos\frac{1}{2}(A+2B-A)} = \frac{\sin B}{\cos B} = \tan B.$$

Ex. 3. Prove that $\dfrac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A} = \dfrac{\sin 3A}{\sin 5A}$,

$$\frac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A} = \frac{(\sin A+\sin 5A)+2\sin 3A}{(\sin 3A+\sin 7A)+2\sin 5A}$$
$$= \frac{2\sin 3A\cos 2A+2\sin 3A}{2\sin 5A\cos 2A+2\sin 5A} = \frac{2\sin 3A\,(1+\cos 2A)}{2\sin 5A\,(1+\cos 2A)} = \frac{\sin 3A}{\sin 5A}.$$

142. The sum or difference of a sine and cosine can be transformed into a product by replacing the sine by the cosine of the complement, or the cosine by the sine of the complement.

Ex. 1. To express $\sin A+\cos B$ as the product of a sine and a cosine.

Looking at (80–83), we see that the expression which transforms into the product of a sine and cosine is a sum or difference of *sines*. We therefore replace the cosine by a sine; thus:

$$\sin A+\cos B = \sin A+\sin(90^\circ-B) = 2\sin\frac{A+90^\circ-B}{2}\cos\frac{A-90^\circ+B}{2}$$
$$= 2\sin\{\tfrac{1}{2}(A-B)+45^\circ\}\cos\{\tfrac{1}{2}(A+B)-45^\circ\}.$$

Ex. 2. To express $\cos A-\sin B$ as the product of two cosines.

The expression which transforms into the product of two cosines is a *sum of cosines*. We therefore take

$$\cos A-\sin B = \cos A-\cos(B-90^\circ) = \cos A+\cos(90^\circ+B)$$
$$= 2\cos\frac{90^\circ+A+B}{2}\cos\frac{90^\circ+B-A}{2}$$
$$= 2\cos\{\tfrac{1}{2}(B+A)+45^\circ\}\cos\{\tfrac{1}{2}(B-A)+45^\circ\}.$$

143. **In deducing other identities from the formulae of this chapter.** it is advisable to observe the following hints:—

(1) In an identity, arrange so as to prove the more complex expression equal to the simpler.

(2) It is *sometimes* useful to expand the trigonometric functions of compound angles; but this should not be done *unnecessarily*, as expressions are often made more unwieldy by so doing.

(3) As a rule, reduce to sines and cosines; but, when only tangents occur, reduce to tangents.

(4) Having simplified one side, if there is no obvious way of equating it to the other, simplify the other likewise.

(5) When both sides are reduced to the same expression, it may be inferred that they are equal.

(6) If expressions involving half-angles, as $\frac{S+T}{2}$, $\frac{S-T}{2}$, occur, it is often easier to put

$$\frac{S+T}{2} = A, \quad \frac{S-T}{2} = B, \quad S = A+B, \quad T = A-B.$$

(7) In simplifying long fractions, it is usually best to express both numerator and denominator as products, because common factors can then be very often cancelled out.

144. **Properties of three angles whose sum is 180°.**—As examples on several of the preceding sections, we give some of the properties of three angles whose sum is 180°.

[This is the relation satisfied by the three angles of a triangle (Euc. I. 32), hence the following results are of importance in connection with the properties of triangles.]

The three angles will be denoted by A, B, C. Since

$$A+B+C = 180^\circ$$
$$\therefore \quad A+B = 180^\circ - C,$$

and, generally, the sum of any two angles is the supplement of the third, so that

$$\sin(A+B) = \sin C, \qquad \cos(A+B) = -\cos C,$$
$$\tan(A+B) = -\tan C, \text{ etc.} \qquad \text{(I.)}$$

Also $\frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C$, and, generally, the semi-sum of any two angles is the complement of half the third, so that

$$\sin\tfrac{1}{2}(A+B) = \cos\tfrac{1}{2}C, \qquad \cos\tfrac{1}{2}(A+B) = \sin\tfrac{1}{2}C,$$
$$\tan\tfrac{1}{2}(A+B) = \cot\tfrac{1}{2}C, \text{ etc.} \qquad \text{(II.)}$$

By using relations such as (I.), (II.) in conjunction with the formulae connecting sums and differences of trigonometric functions with products, and *vice versa*, there will be little difficulty in transforming sums of functions involving A, B, C into products, and *vice versa*, by combining two terms at a time, and then combining the result with the third.

In such transformations it is generally quite indifferent which terms are first combined.

Ex. 1. Express as a product $\sin A+\sin B+\sin C$.

$$\sin A+\sin B+\sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{A+B}{2}\cos\frac{C}{2},$$

[by (II.), since $\frac{1}{2}(A+B)$, and $\frac{1}{2}C$ are complementary]

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 2. Express as a product $\sin A+\sin B-\sin C$.

$$\sin A+\sin B-\sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 3. Prove that $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$.

$$\begin{aligned}\sin 2A+\sin 2B+\sin 2C &= 2\sin(A+B)\cos(A-B)+2\sin C\cos C\\ &= 2\sin C\cos(A-B)-2\sin C\cos(A+B)\\ &= 4\sin C\sin A\sin B.\end{aligned}$$

Ex. 4. Express as a product $\cos A+\cos B+\cos C-1$.

$$\cos A+\cos B+\cos C-1 = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\cos C-1$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}-1,$$

(had $\cos\frac{1}{2}C$ occurred in the first term, we should have written $2\cos^2\frac{1}{2}C-1$ instead of $1-2\sin^2\frac{1}{2}C$ for $\cos C$)

$$= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 5. Express as a product $\cos A+\cos B-\cos C+1$.

$$\cos A+\cos B-\cos C+1 = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-1+2\sin^2\frac{C}{2}+1$$

$$= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right)$$

$$= 4\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}.$$

Ex. 6. Express as a product $\cos 2A+\cos 2B+\cos 2C+1$

$$\cos 2A+\cos 2B = 2\cos(A+B)\cos(A-B) = -2\cos C\cos(A-B),$$

$$\cos 2C+1 = 2\cos^2 C = -2\cos C\cos(A+B);$$

$$\therefore\quad \cos 2A+\cos 2B+\cos 2C+1 = -2\cos C\{\cos(A-B)+\cos(A+B)\}$$

$$= -4\cos A\cos B\cos C.$$

Ex. 7. Express as a product $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}-1$.

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2} = \frac{1-\cos A}{2}+\frac{1-\cos B}{2} = 1-\frac{\cos A+\cos B}{2}$$

$$= 1-\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 1-\sin\frac{C}{2}\cos\frac{A-B}{2},$$

and
$$\sin^2\frac{C}{2}-1 = \sin\frac{C}{2}\cos\frac{A+B}{2}-1;$$

$$\therefore\quad \sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}-1 = -\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)$$

$$= -2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 8. Prove that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

[*This example is very important as a type.*]

$$\tan\frac{A}{2} = \cot\frac{B+C}{2};$$

$$\therefore \quad \tan\frac{A}{2}\tan\frac{B+C}{2} = 1,$$

$$\frac{\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}{1-\tan\frac{B}{2}\tan\frac{C}{2}} = 1;$$

$$\therefore \quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{A}{2}\tan\frac{C}{2} = 1-\tan\frac{B}{2}\tan\frac{C}{2},$$

or
$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

Ex. 9. Prove that $\tan A+\tan B+\tan C = \tan A \tan B \tan C$.

The work is exactly like Ex. 8; we begin by taking the equation

$$\tan A = -\tan(B+C);$$

$$\therefore \quad \tan A = -\frac{\tan B+\tan C}{1-\tan B\tan C};$$

$$\therefore \quad \tan A\,(1-\tan B\tan C) = -\tan B-\tan C;$$

$$\therefore \quad \tan A+\tan B+\tan C = \tan A\tan B\tan C.$$

145. **Symmetry.**—In transforming expressions such as those of the preceding examples, the correctness of the results obtained may always be tested by considerations of symmetry. The same laws apply to trigonometric as to algebraic expressions, and it will be convenient to recapitulate them here.

DEFINITION.—An expression is said to be **symmetrical** in two or more variables when any two of the variables in question can be interchanged without altering the value of the expression.

Thus $\sin(A+B)$ and $\tan A+\tan B$ are symmetrical in A and B; $\cos A\cos B\cos C$ and $\sin(B+C)+\sin(C+A)+\sin(A+B)$ are symmetrical in A, B, and C; and so on.

It is very important to observe that **$\cos(A-B)$ is symmetrical in A and B**, but $\sin(A-B)$ and $\tan(A-B)$ are unsymmetrical.

For $\cos(A-B) = \cos\{-(A-B)\} = \cos(B-A)$,

i.e. = the expression found by interchanging A and B;

$\therefore$ $\cos(A-B)$ is symmetrical.

Since $\sin(B-A) = -\sin(A-B)$ and $\tan(B-A) = -\tan(A-B)$, the sine and tangent of $A-B$ are altered in sign by interchanging A, B, and are therefore unsymmetrical.

On the other hand, $\sin^2(A-B)$ and $\tan^2(A-B)$ are symmetrical in A and B.

COR.—Hence such expressions as $\cos(B-C)+\cos(C-A)+\cos(A-B)$ and $\cos(B-C)\cos(C-A)\cos(A-B)$ are symmetrical in A, B, C.

146. **The Principle of Symmetry** may be stated thus:—*If two expressions are identically equal, and if one of them is symmetrical in any variables, the other will be symmetrical in the same variables.*

Thus $\cos(A-B)$, which is symmetrical in A and B, is equal to $\cos A \cos B+\sin A \sin B$, which is also symmetrical in A and B.

The principle is applicable whether the variables are independent or are connected by any *symmetrical* relation, such as the condition $A+B+C = 180°$ of the last article.

Thus, in § 144, Ex. 1, the symmetrical expression $\sin A+\sin B+\sin C$ is equal to another symmetrical expression, viz. $4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$. If we had found an unsymmetrical result, such as $4\cos\frac{1}{2}A\cos\frac{1}{2}B\sin\frac{1}{2}C$ we should have concluded that there was a mistake in our work.

Again, in § 144, Ex. 2, $\sin A+\sin B-\sin C$ is symmetrical in A and B *only*, and, therefore, the same must be true of the expression to which it is to be proved equal, viz. $4\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C$, and this is actually seen to be the case. Had we obtained as our result *either* $4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ *or* $4\sin\frac{1}{2}A\cos\frac{1}{2}B\sin\frac{1}{2}C$ we should have inferred that the result was incorrect, the first because of its *being* symmetrical in A, B, *and* C instead of only in A and B; the second because of its *not* being symmetrical in A and B.

147. In certain cases the Principle of Symmetry may be used to establish identities.

Thus in § 144, Ex. 5, we proved that, if $A+B+C = 180°$,

$$\cos A+\cos B-\cos C+1 = 4\sin\tfrac{1}{2}C\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B \quad \text{(i)}$$

Adding $2\cos C$ to both sides, we obtain

$$\cos A+\cos B+\cos C+1 = 2\cos C+4\sin\tfrac{1}{2}C\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B.$$

The left-hand side is symmetrical in A, B, C; therefore the right-hand side must also be symmetrical in A, B, C, that is, its value must be unaltered when the letters A, B, C are interchanged in any way (taking account, of course, of the identical relation $A+B+C = 180°$).

By interchanging the letters and equating the different expressions, we therefore obtain the identity

$$\cos A+2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \cos B+2\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C\cos\tfrac{1}{2}A$$
$$= \cos C+2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B.$$

Again, by multiplying (i) by $\cot\frac{1}{2}C$, the right-hand side becomes symmetrical, giving

$$\cot\tfrac{1}{2}C(\cos A+\cos B-\cos C+1) = 4\cos\tfrac{1}{2}C\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B.$$

We conclude that the left-hand side is also symmetrical, and hence that

$$\cot\tfrac{1}{2}A(\cos B+\cos C-\cos A+1) = \cot\tfrac{1}{2}B(\cos C+\cos A-\cos B+1)$$
$$= \cot\tfrac{1}{2}C(\cos A+\cos B-\cos C+1).$$

147*a*. To prove the sum and difference formulae by geometry.

In Fig. 99*a* let $\angle\mathbf{POX} = S$,

and $\angle\mathbf{QOX} = T$.

Let **OR** bisect $\angle\mathbf{POQ}$, and through **R** draw **PRQ** perpendicular to **OR**.

Then

$$\angle\mathbf{POR} = \angle\mathbf{ROQ} = \tfrac{1}{2}(S-T);$$
$$\angle\mathbf{ROX} = \angle\mathbf{ROQ}+\angle\mathbf{QOX}$$
$$= \tfrac{1}{2}(S-T)+T = \tfrac{1}{2}(S+T).$$

Also $\triangle\,\mathbf{OPR} \equiv \triangle\,\mathbf{OQR}$.

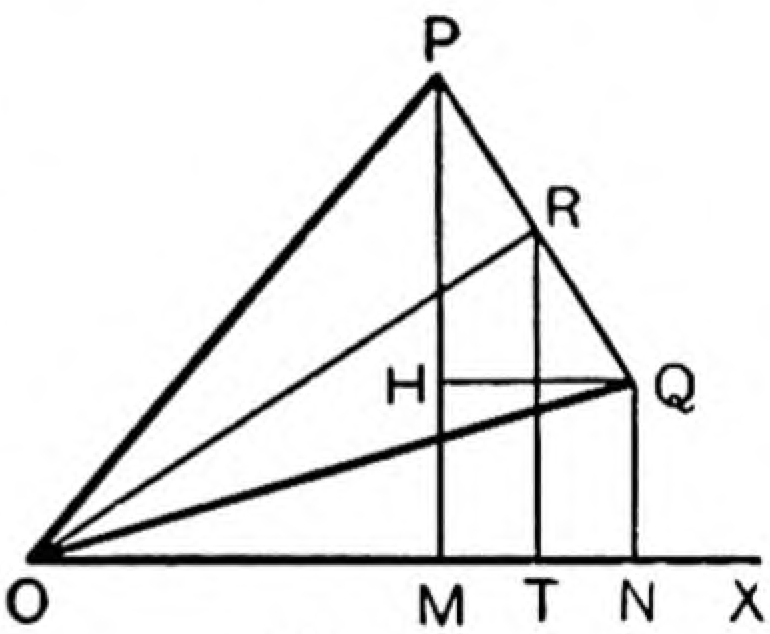


Fig. 99*a*.

Draw **PM, RT, QN** perpendicular to **OX**, and **QH** perpendicular to **PM**.

Since the arms of $\angle\mathbf{QPH}$ are respectively perpendicular to those of $\angle\mathbf{ROX}$, therefore

$$\angle\mathbf{QPH} = \angle\mathbf{ROX} = \tfrac{1}{2}(S+T).$$

Again, since **R** is the middle point of **PQ** and **PM, QN, RT** are all parallel.

$$\mathbf{TM} = \mathbf{TN},$$
$$\mathbf{OT} = \mathbf{ON}-\mathbf{TN} = \mathbf{OM}+\mathbf{MT}$$
$$= \tfrac{1}{2}\{\mathbf{ON}-\mathbf{TN}+\mathbf{OM}+\mathbf{MT}\} = \tfrac{1}{2}(\mathbf{ON}+\mathbf{OM})$$
$$\mathbf{RT} = \tfrac{1}{2}(\mathbf{PM}+\mathbf{QN}).$$

Let $\mathbf{OP} = \mathbf{OQ} = l.$

(i) $l\sin S+l\sin T = \mathbf{PM}+\mathbf{QN}$

$= 2\mathbf{RT}$

$= 2\mathbf{OR}\sin\mathbf{ROT} = 2\mathbf{OP}\cos\mathbf{POR}\sin\mathbf{ROT}$

$= 2l\cos\tfrac{1}{2}(S-T)\sin\tfrac{1}{2}(S+T)$

or $\sin S+\sin T = 2\sin\tfrac{1}{2}(S+T)\cos\tfrac{1}{2}(S-T)$.

(ii) $l\sin S-l\sin T = \mathbf{PM}-\mathbf{QN}$

$= \mathbf{PH}$

$= \mathbf{PQ}\cos\mathbf{QPH}$

$= 2\mathbf{PR}\cos\mathbf{QPH}$

$= 2\mathbf{OP}\sin\mathbf{POR}\cos\mathbf{QPH}$

$= 2l\sin\tfrac{1}{2}(S-T)\cos\tfrac{1}{2}(S+T)$

or $\sin S-\sin T = 2\sin\tfrac{1}{2}(S-T)\cos\tfrac{1}{2}(S+T)$.

(iii) $l\cos S+l\cos T = \mathbf{OM}+\mathbf{ON}$

$= 2\mathbf{OT}$

$= 2\mathbf{OR}\cos\mathbf{ROT}$

$= 2\mathbf{OP}\cos\mathbf{POR}\cos\mathbf{ROT}$

$= 2l\cos\tfrac{1}{2}(S-T)\cos\tfrac{1}{2}(S+T)$

or $\cos S+\cos T = 2\cos\tfrac{1}{2}(S+T)\cos\tfrac{1}{2}(S-T)$.

(iv) $l\cos T-l\cos S = \mathbf{ON}-\mathbf{OM} = \mathbf{MN}$

$= 2\mathbf{MT}$

$= 2\mathbf{PR}\sin\mathbf{QPH}$

$= 2\mathbf{OP}\sin\mathbf{POR}\sin\mathbf{QPH}$

$= 2l\sin\tfrac{1}{2}(S-T)\sin\tfrac{1}{2}(S+T)$

or $\cos T-\cos S = 2\sin\tfrac{1}{2}(S+T)\sin\tfrac{1}{2}(S-T)$.

EXAMPLES XIII.

1. Prove the formula $\cos A+\cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$, and investigate the corresponding formula for $\cos A-\cos B$.

2. Prove, geometrically, that $\sin A+\sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$, when each of the angles A and B is less than a right angle.

FIND the simplest forms of the following expressions (3–7):—

3. $\dfrac{\sin 7\theta-\sin 5\theta}{\cos 7\theta-\cos 5\theta}$.

4. $\dfrac{\cos 6\theta-\cos 4\theta}{\sin 6\theta+\sin 4\theta}$.

5. $\dfrac{\sin 3\theta+\sin 5\theta-\sin 4\theta}{\cos 3\theta+\cos 5\theta-\cos 4\theta}$.

6. $\dfrac{\sin A+\sin 2A+\sin 3A+\sin 4A}{\cos A+\cos 2A+\cos 3A+\cos 4A}$.

7. $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}$.

8. Find the values of θ between which $\cos 3\theta-\cos\theta$ has positive values, and also those between which $\sin 3\theta-\sin\theta$ has positive values.

9. Show that $\sin\left(\dfrac{3\pi}{2}-\theta\right)+\cos\theta=0$, where θ is any angle.

10. If $\tan\dfrac{\theta}{2}\tan\dfrac{\phi}{2}=\tan^2\dfrac{a}{2}$, prove that

(a) $\cos\dfrac{\theta+\phi}{2}=\cos\dfrac{\theta-\phi}{2}\cos a$, (b) $\dfrac{\sin^2\dfrac{\theta-a}{2}}{\sin^2\dfrac{\theta+a}{2}}=\dfrac{\sin\left(\dfrac{\theta+\phi}{2}-a\right)}{\sin\left(\dfrac{\theta+\phi}{2}+a\right)}$.

11. Prove that $(\cos x-\cos y)^2+(\sin x-\sin y)^2=4\sin^2\frac{1}{2}(x-y)$.

12. If $\cos(x+y)=\cos z$, show that

$$1-\cos^2 x-\cos^2 y-\cos^2 z+2\cos x\cos y\cos z=0.$$

13. If $(\cos^2 x-\sin^2 y)\tan^2 z-\sin^2 x=0$, show that

$$\sin x=\pm\sin z\cos y.$$

PROVE the following identities (14–51):—

14. $\dfrac{\sin 7A+\sin 3A}{\cos 7A+\cos 3A}=\tan 5A$.

15. $\dfrac{\sin 3A-\sin A}{\cos A-\cos 3A}=\cot 2A$.

16. $\dfrac{\sin 4A-\sin 2A}{\cos 2A-\cos 4A}=\cot 3A$.

17. $\dfrac{\cos A-\cos 3A}{\sin 3A-\sin A}=\tan 2A$.

18. $\dfrac{\sin 6A+\sin 8A}{\cos 10A+\cos 4A}=\dfrac{\tan 7A\cos A}{\cos 3A}$.

19. $\dfrac{\sin 5A-\sin 3A}{\cos 3A-\cos 5A}=\cot 4A$.

20. $\dfrac{\sin(2A+3B)+\sin(3A+2B)}{\cos(2A+3B)+\cos(3A+2B)}=\tan\dfrac{5(A+B)}{2}$.

21. $\dfrac{\cos 7A-\cos 9A}{\sin 9A-\sin 7A}=\tan 8A$.

22. $\dfrac{2\sin(A-B)\cos B-\sin(A-2B)}{2\sin(C-B)\cos B-\sin(C-2B)}=\dfrac{\sin A}{\sin C}$.

23. $\dfrac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A$.

24. $\dfrac{\sin 5A+\sin 9A}{\cos 5A-\cos 9A}=\cot 2A$.

25. $\dfrac{\cos 37^\circ+\sin 37^\circ}{\cos 37^\circ-\sin 37^\circ} = \cot 8^\circ.$

26. $\dfrac{\cos A-\cos 3A}{\sin 9A-\sin 7A} = \dfrac{\sin 2A}{\cos 8A}.$

27. $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ = 0.$

28. $\sin 85^\circ = \cos 55^\circ+\sin 25^\circ.$

29. $\cos 5^\circ-\sin 25^\circ = \sin 35^\circ.$

30. $\sin 50^\circ-\sin 70^\circ+\sin 10^\circ = 0.$

31. $\cos (2n-1) A\pm\cos nA+\cos A$
$$= \cot nA\{\sin (2n-1)A\pm\sin nA+\sin A\}.$$

32. $\sin (A+B+C)+\sin (A+B-C)+\sin (A-B+C)+\sin (A-B-C)$
$$= 4 \sin A \cos B \cos C.$$

33. $\sin A (\sin 2A+\sin 4A+\sin 6A) = \sin 3A \sin 4A.$

34. $\sin A (\cos 2A+\cos 4A+\cos 6A) = \sin 3 A \cos 4A.$

35. $\sin A \cos (A+B)-\cos A \sin (A-B) = \cos 2A \sin B.$

36. $\sin (A-B) \cos 2B+\cos (A-B) \sin 2B$
$$= \sin (B-A) \cos 2A+ \cos (B-A) \sin 2 A$$

37. $\dfrac{\sin A}{\cos B}-\dfrac{\sin B}{\cos A} = \dfrac{2 \sin (A-B) \cos (A+B)}{\cos (A+B)+\cos (A-B)}.$

38. $\dfrac{\sin A \sin 2A+\sin 2A \sin 5A+ \sin 3A \sin 10A}{\sin A \cos 2A+\sin 2A \cos 5A+\sin 3A \cos 10A} = \tan 7A.$

39. $\dfrac{\sin A \sin 2A+\sin 3A \sin 6A+\sin 4A \sin 13A}{\sin A \cos 2A+\sin 3A \cos 6A+\sin 4A \cos 13A} = \tan 9A.$

40. $\sin (A+B+C) \sin B = \sin (A+B) \sin (B+C)-\sin A \sin C.$

41. $\cos A-\cos B-\sin (A-B)$
$$= 2 \sin \frac{B-A}{2} \left(\sin \frac{A}{2} +\cos \frac{A}{2}\right) \left(\sin\frac{B}{2} +\cos\frac{B}{2}\right).$$

42. $\cos A+\cos B-\sin (A+B)$
$$= 2 \cos \frac{A+B}{2} \left(\cos \frac{A}{2} -\sin\frac{A}{2}\right) \left(\cos\frac{B}{2} -\sin\frac{B}{2}\right).$$

43. $\dfrac{\sin (A-B)+\sin A+\sin (A+B)}{\sin (C-B)+\sin C+\sin (C+B)} = \dfrac{\sin A}{\sin C}.$

44. $\dfrac{1-\cos A-\cos (A+C)+\cos C}{1-\cos C-\cos (A+C)+\cos A} = \tan \dfrac{A}{2}\cot \dfrac{C}{2}.$

45. $\sin 5A+\sin 7A = 4 \sin A \cos^2 A (1+2 \cos 4A).$

46. $\sin A+\sin B-\sin C+\sin (A+B+C)$
$$= 4 \sin \frac{A+B}{2} \cos \frac{A+C}{2} \cos\frac{B+C}{2}.$$

47. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}.$

48. $\cos\frac{\pi}{16}+\cos\frac{3\pi}{16}+\cos\frac{5\pi}{16}+\cos\frac{7\pi}{16}=\frac{1}{2}\operatorname{cosec}\frac{15\pi}{16}.$

49. $\operatorname{cosec}\frac{\pi}{2}+\operatorname{cosec}\frac{\pi}{4}+\operatorname{cosec}\frac{\pi}{8}=\cot\frac{\pi}{16}.$

50. $\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}\left(\operatorname{cosec}\frac{\pi}{14}-1\right).$

51. $$\cos\frac{\alpha+\beta+\gamma}{2}+\cos\frac{3\alpha-\beta-\gamma}{2}+\cos\frac{3\beta-\gamma-\alpha}{2}+\cos\frac{3\gamma-\alpha-\beta}{2}$$
$$=4\cos\frac{\beta+\gamma-\alpha}{2}\cos\frac{\gamma+\alpha-\beta}{2}\cos\frac{\alpha+\beta-\gamma}{2}.$$

If A, B, C, are the angles of a triangle, prove the following relations (52–62):—

52. $\tan A+\tan B+\tan C=\tan A\tan B\tan C.$

53. $\cot A\cot B+\cot B\cot C+\cot C\cot A=1.$

54. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

55. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

56. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$

57. $\cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C+1=0.$

58. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

59. $$\sin A\sin B\sin C=\sin A\cos B\cos C+\sin B\cos A\cos C$$
$$+\sin C\cos A\cos B.$$

60. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{B+C}{4}\sin\frac{C+A}{4}\sin\frac{A+B}{4}.$

61. $(\sin A+\sin B-\sin C)\tan\frac{C}{2}=(\sin B+\sin C-\sin A)\tan\frac{A}{2}.$

62. $\sin 3A+\sin 3B+\sin 3C+4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}=0.$

63. If $\alpha+\beta+\gamma=2\pi$, prove that
$$\sin\beta(1+2\cos\gamma)+\sin\gamma(1+2\cos\alpha)+\sin\alpha(1+2\cos\beta)$$
$$=4\sin\frac{\gamma-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\alpha}{2}.$$

CHAPTER XIV.

EQUATIONS AND INVERSE FUNCTIONS INVOLVING DIFFERENT ANGLES.

148. We shall now illustrate the results established in the preceding three chapters by applying them to the solution of trigonometrical equations involving multiple angles and the properties of two or more inverse functions.

149. To find sin 18° and cos 18°.

Let $A = 18°$. Then $5A = 90°$; hence $2A = 90° - 3A$;

$$\therefore \quad \sin 2A = \sin(90° - 3A) = \cos 3A;$$

$$\therefore \quad 2\sin A\cos A = 4\cos^3 A - 3\cos A.$$

Since $\cos A$ is a factor of both sides, one solution is $\cos A = 0$, corresponding to $A = 90°$, $270°$, etc. It is easy to verify that these angles as well as $18°$ satisfy the equation $\sin 2A = \cos 3A$, for instance, $\sin 2(90°) = 0 = \cos 3(90°)$; these solutions are, however, irrelevant to the present problem (see Ex. 2 below).

Dividing out by $\cos A$, the remaining solutions are given by

$$2\sin A = 4\cos^2 A - 3 = 4 - 4\sin^2 A - 3 = 1 - 4\sin^2 A;$$

$$\therefore \quad 4\sin^2 A + 2\sin A - 1 = 0.$$

Solving this as a quadratic in $\sin A$, we have

$$\sin A = \frac{-2 \pm \sqrt{\{2^2 - 4.4.(-1)\}}}{2.4} = \frac{-2 \pm \sqrt{20}}{8}$$

$$= \frac{-2 \pm 2\sqrt{5}}{8} = \frac{\pm\sqrt{5} - 1}{4}.$$

Hence sin 18° must have *one or other* of two values

$$\frac{\sqrt{5}-1}{4}, \quad \frac{-\sqrt{5}-1}{4}.$$

But sin 18° is evidently positive. Therefore

$$\mathbf{\sin 18^\circ = \frac{\sqrt{5}-1}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(84)}$$

The value of cos 18° is most easily deduced (*whenever required**) from that of sin 18°, thus

$$\cos^2 18^\circ = 1-\sin^2 18^\circ = 1-\frac{5-2\sqrt{5}+1}{16} = \frac{10+2\sqrt{5}}{16};$$

$$\therefore \quad \cos 18^\circ = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

Cor.—Since $72^\circ = 90^\circ-18^\circ$, it follows that

$$\cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}, \text{ and } \sin 72^\circ = \cos 18^\circ$$

$$= \frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

Ex. 1. To find the sine and cosine of 36° and 54°.

$$\sin 54^\circ = \cos 36^\circ = 1-2\sin^2 18^\circ = 1-\frac{5-2\sqrt{5}+1}{8}$$

$$= \frac{2+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4},$$

$$\cos 54^\circ = \sin 36^\circ = \sqrt{(1-\cos^2 36^\circ)} = \sqrt{\left\{1-\frac{5+2\sqrt{5}+1}{16}\right\}}$$

$$= \frac{\sqrt{(10-2\sqrt{5})}}{4}.$$

We have now found the sines and cosines of 18° and its successive multiples 36°, 54°, and 72°. The next multiple is 90°, and higher multiples are treatable by the methods of Chap. VIII., thus

$$108^\circ = 90^\circ+18^\circ, \text{ etc.}$$

Ex. 2. To find a general expression for *all* the angles which satisfy the equation of the present article, viz.

$$\sin 2\theta = \cos 3\theta.$$

---

* The value need not be remembered.

We write the equations thus:

$$\cos 3\theta = \cos(\tfrac{1}{2}\pi - 2\theta);$$

$$\therefore \quad 3\theta = 2n\pi \pm (\tfrac{1}{2}\pi - 2\theta);$$

$$\therefore \quad \theta = \frac{2n \pm \frac{1}{2}}{3 \pm 2}\pi = \frac{4n \pm 1}{6 \pm 4}\pi,$$

the upper or lower sign being taken in both the numerator and denominator. Taking the upper sign and putting $n = 0, 1, 2, \ldots$, we have

$$\theta = \tfrac{1}{10}\pi,\ \tfrac{1}{2}\pi,\ \tfrac{9}{10}\pi,\ 1\tfrac{3}{10}\pi,\ 1\tfrac{7}{10}\pi,\ 2\tfrac{1}{10}\pi,\ 2\tfrac{1}{2}\pi, \ldots.$$

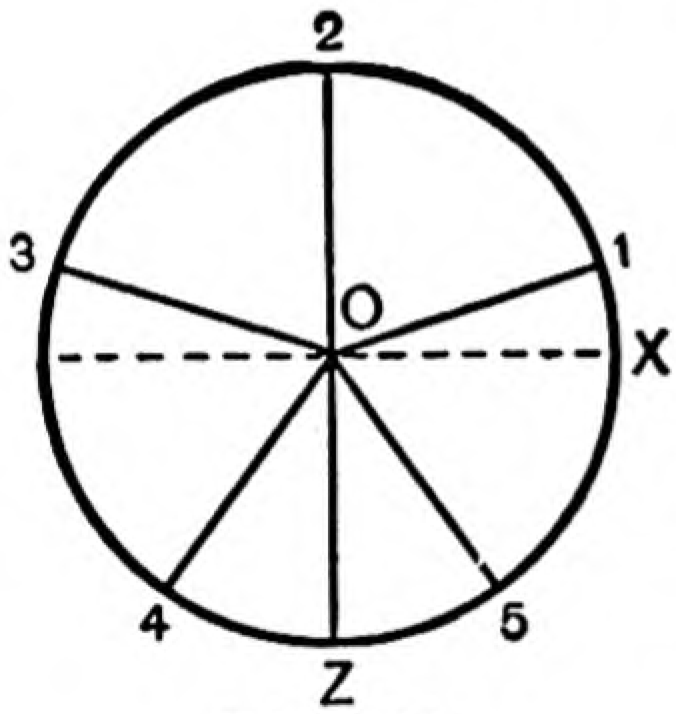


Fig. 100.

If the angles be represented in a figure (Fig. 100), we obtain the angles **XO1**, **XO2**, **XO3**, **XO4**, **XO5**, and angles coterminal with them, the common difference of successive angles being $\tfrac{2}{5}\pi$.

Of these, the first $= 18°$, the third has its sine $= \sin 18° = \tfrac{1}{4}(\sqrt{5}-1)$, the fourth and fifth have their sines each $= -\sin 54° = -\tfrac{1}{4}(\sqrt{5}+1)$ (the second root of the equation of § 146), and the second satisfies the equation $\cos\theta = 0$, which we rejected as irrelevant.

Taking the lower sign in the ambiguity, we have the series of angles

$$\theta = -\tfrac{1}{2}\pi,\ 1\tfrac{1}{2}\pi,\ 3\tfrac{1}{2}\pi,\ 5\tfrac{1}{2}\pi, \ldots,$$

all of which are coterminal with **XOZ** and satisfy the equation

$$\cos\theta = 0.$$

150. Knowing the sine and cosine of 18° and also those of 15°, we can at once deduce sin 3° and cos 3°, and then we can find the sine and cosine of any multiple of 3°. Also the "$\tfrac{1}{2}A$ formulae" enable us to find the sines and cosines of such submultiples of 3° as $1\tfrac{1}{2}°$, $\tfrac{3}{4}°$, $\tfrac{3}{8}°$ ..., and their multiples.

All such functions are representable by surd formulae involving nothing more than square roots, but these formulae are too complicated to be of much use except in a few simple cases.

Ex. 1. To find sin 3°.

$$\sin 3° = \sin(18° - 15°) = \sin 18° \cos 15° - \cos 18° \sin 15°$$

$$= \frac{\sqrt{2}(\sqrt{3}+1)(\sqrt{5}-1)}{16} - \frac{(\sqrt{3}-1)\sqrt{(5+\sqrt{5})}}{8}.$$

Ex. 2. To find the sine and cosine of 9°.

$$\sin 9° = \sin(54°-45°) = \frac{\sqrt{5}+1}{4}\frac{1}{\sqrt{2}} - \frac{\sqrt{(10-2\sqrt{5})}}{4}\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{10}+\sqrt{2}}{8} - \frac{\sqrt{(5-\sqrt{5})}}{4};$$

similarly,
$$\cos 9° = \frac{\sqrt{10}+\sqrt{2}}{8} + \frac{\sqrt{(5-\sqrt{5})}}{4}.$$

151. Trigonometric equations involving multiple angles.

As facility in solving equations comes by practice we have thought it advisable to work a considerable number of illustrative examples as types. The student in solving similar equations is recommended to be on the look out for *any legitimate artifice* by means of which the work may be shortened.

Ex. 1. Solve
$$\sin 7\alpha - \sin\alpha = \sin 3\alpha;$$

$$\therefore\quad 2\cos 4\alpha \sin 3\alpha = \sin 3\alpha;$$

$$\therefore\quad \text{either } \sin 3\alpha = 0, \text{ or } \cos 4\alpha = \tfrac{1}{2};$$

[It must never be forgotten that, whenever any factor cancels out, a root or set of roots will be found by equating it to zero.]

$$\therefore\quad 3\alpha = n\pi, \text{ or } 4\alpha = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore\quad \alpha = \frac{n\pi}{3}, \text{ or } \alpha = \frac{1}{4}\left(2n\pi \pm \frac{\pi}{3}\right);$$

of which particular solutions are

$$\alpha = 0°, 60°, \ldots, \text{ or } \alpha = 15°, \ldots, \text{etc.}$$

Ex. 2. Solve
$$\sin 2\theta = \cos\theta;$$

$$\therefore\quad 2\sin\theta\cos\theta = \cos\theta;$$

$$\therefore\quad \text{either } \cos\theta = 0, \quad \text{or } \sin\theta = \tfrac{1}{2} = \sin\frac{\pi}{6};$$

$$\therefore\quad \theta = 2n\pi \pm \frac{\pi}{2}, \text{ or } = n\pi + (-1)^n\frac{\pi}{6};$$

of which particular solutions are

$$\theta = 90°, \ldots, \text{ or } \theta = 30°, 150°, \ldots, \text{etc.}$$

Ex. 3. Solve
$$\sin 5x \cos 3x = \sin 9x \cos 7x;$$

$$\therefore\quad \tfrac{1}{2}(\sin 8x + \sin 2x) = \tfrac{1}{2}(\sin 16x + \sin 2x);$$

$$\therefore\quad \sin 8x = \sin 16x = 2\sin 8x\cos 8x;$$

$$\therefore \quad \text{either } \sin 8x = 0, \quad \text{or } \cos 8x = \tfrac{1}{2};$$

$$\therefore \quad 8x = n\pi, \quad \text{or } \; = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \quad x = \frac{n\pi}{8}, \quad \text{or } \; = \frac{n\pi}{4} \pm \frac{\pi}{24};$$

of which particular solutions are

$$8x = 0°, 180°, 360°, \ldots, \quad \text{or } \; 8x = 60°, 300°, \ldots, \text{ etc.};$$

$$\therefore \quad x = 0°, 22° 30', 45°, \ldots, \quad \text{or } \; x = 7° 30', 37° 30', \text{ etc.}$$

Ex. 4. Solve $\quad \sin 6\theta = \sin 4\theta - \sin 2\theta;$

$$\therefore \quad \sin 4\theta = \sin 6\theta + \sin 2\theta = 2 \sin 4\theta \cos 2\theta;$$

$$\therefore \quad \text{either } \sin 4\theta = 0, \quad \text{or } \cos 2\theta = \tfrac{1}{2};$$

$$\therefore \quad 4\theta = n\pi, \quad \text{or } \; 2\theta = 2n\pi \pm \frac{\pi}{3};$$

$$\theta = \frac{n\pi}{4}, \quad \text{or } \; = n\pi \pm \frac{\pi}{6},$$

of which particular solutions are

$$4\theta = 0°, 180°, 360°, \ldots, \quad \text{or } \; 2\theta = 60°, 300°, \ldots, \text{ etc.};$$

$$\theta = 0°, 45°, 90°, \ldots, \quad \text{or } \; \theta = 30°, 150°, \ldots, \text{ etc.}$$

Illustrative Exercise.

Illustrate each of the above examples by a diagram, representing *all* the angles which satisfy the given equation.

152. **General expressions.**—The following method of arriving at the results proved in §§ 104–106 is instructive:

Ex. 1. Solve for θ the equation $\cos \theta = \cos \alpha$.

Writing the equation $\quad \cos \alpha - \cos \theta = 0,$

we have $\quad 2 \sin \tfrac{1}{2} (\theta - \alpha) \sin \tfrac{1}{2} (\theta + \alpha) = 0;$

$$\therefore \quad \text{either } \sin \tfrac{1}{2} (\theta - \alpha) = 0, \quad \text{or } \sin \tfrac{1}{2} (\theta + \alpha) = 0;$$

$\therefore$ either $\tfrac{1}{2} (\theta - \alpha) =$ a multiple of two right angles $= n\pi$,

or $\tfrac{1}{2} (\theta + \alpha) =$,, ,, ,, $= n\pi$;

$$\therefore \quad \text{either } \theta = 2n\pi + \alpha, \quad \text{or } \theta = 2n\pi - \alpha,$$

that is $\quad \theta = 2n\pi \pm \alpha,$

where n is zero or a positive or negative integer, as in § 105.

Ex. 2. Solve $\sin\theta = \sin\alpha.$

Here $\sin\theta - \sin\alpha = 0$

$$\therefore\quad 2\cos\tfrac{1}{2}(\theta+\alpha)\sin\tfrac{1}{2}(\theta-\alpha) = 0;$$

$$\therefore\quad \text{either } \sin\tfrac{1}{2}(\theta-\alpha) = 0, \text{ or } \cos\tfrac{1}{2}(\theta+\alpha) = 0;$$

$$\therefore\quad \text{either } \tfrac{1}{2}(\theta-\alpha) = m\pi, \quad \text{or } \tfrac{1}{2}(\theta+\alpha) = (m+\tfrac{1}{2})\pi;$$

$$\therefore\quad \theta = 2m\pi+\alpha, \text{ or } \theta = (2m+1)\pi-\alpha.$$

To deduce the general expression $\theta = n\pi+(-1)^n\alpha$, we must now proceed as in § 104.

Otherwise thus :—Writing the equation

$$\cos(\theta-\tfrac{1}{2}\pi) = \cos(\alpha-\tfrac{1}{2}\pi),$$

the general solution is, by Ex. 1,

$$\theta-\tfrac{1}{2}\pi = 2n\pi\pm(\alpha-\tfrac{1}{2}\pi)$$

$$\theta = (2n+\tfrac{1}{2})\pi\pm(\tfrac{1}{2}\pi-\alpha),$$

agreeing with the formula given in § 104, Ex. 2,

Ex. 3. $\tan\theta = \tan\alpha.$

Here $$\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha};$$

$$\therefore\quad \sin\theta\cos\alpha-\cos\theta\sin\alpha = 0, \quad \text{or} \quad \sin(\theta-\alpha) = 0,$$

$$\therefore\quad \theta-\alpha = n\pi, \quad \theta = n\pi+\alpha,$$

as in § 106.

153. Solution of the general linear equation in sin θ and cos θ.

We shall now illustrate a method of solving for θ any equation of the form

$$a\cos\theta\pm b\sin\theta = c$$

without reducing it to quadratic form. The rules illustrated by the following example are general, and the method is especially useful when b/a is the tangent of a well-known angle such as 30°, 45°, 60°, etc.

Ex. 1. Solve for θ the equation

$$3\cos\theta+\sqrt{3}\sin\theta = \sqrt{6}.$$

1st. Divide by the coefficient of $\cos\theta$, thus

$$\cos\theta+\tfrac{1}{3}\sqrt{3}\sin\theta = \tfrac{1}{3}\sqrt{6}.$$

2nd. Express the new coefficient of $\sin\theta$ as a tangent.

Since $\tfrac{1}{3}\sqrt{3} = \tan 30° = \tan\tfrac{1}{6}\pi$, we write the equation

$$\cos\theta+\tan\tfrac{1}{6}\pi\sin\theta = \tfrac{1}{3}\sqrt{6}.$$

3rd. Multiply by the corresponding cosine, viz. $\cos \frac{1}{6}\pi$, and substitute its numerical value on the side on which θ does not occur.

Thus $\cos\theta\cos\frac{1}{6}\pi+\sin\theta\sin\frac{1}{6}\pi = \frac{1}{3}\sqrt{6}.\cos\frac{1}{6}\pi = \frac{1}{3}\sqrt{6}\times\frac{1}{2}\sqrt{3}$;
that is, $$\cos(\theta-\tfrac{1}{6}\pi) = \tfrac{1}{2}\sqrt{2} = \cos\tfrac{1}{4}\pi.$$

4th. Solving this equation, we have
$$\theta-\tfrac{1}{6}\pi = 2n\pi\pm\tfrac{1}{4}\pi;$$
$$\therefore\quad \theta = (2n+\tfrac{1}{6})\,\pi\pm\tfrac{1}{4}\pi,$$
and this is the required general solution.

Linear equations of the forms
$$a\sec\theta+b\tan\theta = c,\quad\text{and}\quad a\operatorname{cosec}\theta+b\cot\theta = c,$$
can be reduced to a similar form and solved in the same manner.

Ex. 2. Solve $$\sqrt{2}\sec\theta+\tan\theta = 1,$$
on multiplying by $\cos\theta$, becomes
$$\sqrt{2}+\sin\theta = \cos\theta\quad\text{or}\quad\cos\theta-\sin\theta = \sqrt{2}.$$

The coefficient of $\cos\theta$ being unity, that of $\sin\theta = -1 = -\tan\frac{1}{4}\pi$;
$$\therefore\quad \cos\theta-\tan\tfrac{1}{4}\pi\sin\theta = \sqrt{2}.$$

Multiplying by $\cos\frac{1}{4}\pi$, we have
$$\cos\theta\cos\tfrac{1}{4}\pi-\sin\theta\sin\tfrac{1}{4}\pi = \sqrt{2}\cos\tfrac{1}{4}\pi = \sqrt{2}\div\sqrt{2} = 1;$$
$$\therefore\quad \cos(\theta+\tfrac{1}{4}\pi) = 1,\quad\text{whence}\quad \theta+\tfrac{1}{4}\pi = 2n\pi;$$
$$\therefore\quad \text{the general solution is}\quad \theta = (2n-\tfrac{1}{4})\,\pi.$$

154. To trace the variations of the expression
$$a\cos\theta + b\sin\theta$$
for different values of θ.

[The method is a slight modification of that employed in solving the equations of the last article.]

Transform the expression thus:
$$a\cos\theta+b\sin\theta = a(\cos\theta+b/a\sin\theta).$$

Let $\alpha = \tan^{-1}b/a$. Since the tangent of an angle may have any value whatever, a real angle α can always be found satisfying this condition, and then, by § 77, Ex. 2, or the method of § 79,
$$\sin\alpha = \frac{b}{\sqrt{(a^2+b^2)}},\quad \cos\alpha = \frac{a}{\sqrt{(a^2+b^2)}}.$$

The given expression now becomes
$$= a(\cos\theta+\tan\alpha\sin\theta) = \frac{a}{\cos\alpha}(\cos\theta\cos\alpha+\sin\theta\sin\alpha)$$
$$= \frac{a}{\cos\alpha}\cos(\theta-\alpha) = \sqrt{(a^2+b^2)}\cos(\theta-\alpha).$$

Hence the variations of the given expression depend on the variations of $\cos(\theta-\alpha)$, and these have been traced in Chap. V.

Thus the maximum value of $\cos(\theta-\alpha)$ is 1 and occurs when $\theta-\alpha=0$.

Hence the maximum value of the given expression is $\sqrt{(a^2+b^2)}$ and occurs when $\theta=\alpha=\tan^{-1} b/a$.

As θ increases from α to $\alpha+\frac{1}{2}\pi$ the given expression decreases from $\sqrt{(a^2+b^2)}$ to 0.

As θ increases from $\alpha+\frac{1}{2}\pi$ to $\alpha+\pi$ the given expression is negative and decreases from 0 to its numerically greatest negative value $-\sqrt{(a^2+b^2)}$, and so on.

COR. 1.—The algebraical maximum and minimum values of $a\cos\theta+b\sin\theta$ are $\sqrt{(a^2+b^2)}$ and $-\sqrt{(a^2+b^2)}$ respectively, and where these values occur θ satisfies the relation $\tan\theta=b/a$.

COR. 2.—In the course of the above investigation we have incidentally established the identity

$$a\cos\theta+b\sin\theta=\sqrt{(a^2+b^2)}\cos\{\theta-\tan^{-1}b/a\}.$$

155. Inverse formulae.—A great many of the identities of the last three chapters can very readily be expressed in the inverse notation.

Ex. 1. To express the formulae for $\sin(A+B)$ and $\sin(A-B)$ in inverse notation.

Putting $\sin A=x$, $\sin B=y$ and remembering the identity $\sin^2+\cos^2=1$, we have $\cos A=\sqrt{(1-x^2)}$, $\cos B=(1-y^2)$ and the formulae give

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\{x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}\},$$
$$\sin^{-1}x-\sin^{-1}y=\sin^{-1}\{x\sqrt{(1-y^2)}-y\sqrt{(1-x^2)}\}.$$

Ex. 2. Similarly, putting $\cos A=x$, $\cos B=y$, in the formulae for $\cos(A+B)$ and $\cos(A-B)$, we have

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{(1-x^2)}\sqrt{(1-y^2)}\},$$
$$\cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{(1-x^2)}\sqrt{(1-y^2)}\}.$$

The formulae of Exs. 1, 2, are but little used, and difficulties arise in connection with them from the fact that expressions such as $\sin^{-1}x$ *may* represent not one angle, but a series of angles; moreover, the radicals may be taken either positive or negative.

It is always easier to work with inverse *tangents*, as the formulae (85, 86) below do not then involve radicals. We can always reduce any inverse function to an inverse tangent by the method of § 100 or by the table of § 102; thus, *e.g.* $\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$.

[N.B.—Always bear in mind that $\sin^{-1}x$, $\cos^{-1}x$, etc., are *not* trigonometrical ratios, but angles.]

156. To prove that

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \quad \text{.............(85)}$$

Let $\tan^{-1} x = A$, $\tan^{-1} y = B$. Then $x = \tan A$, $y = \tan B$, and $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A \tan B} = \dfrac{x+y}{1-xy}$;

$$\therefore \quad A+B \text{ or } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}.$$

Similarly, $$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy} \quad \text{.............(86)}$$

Cor.—Putting $x = y$ in (85), we have

$$2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \quad \text{....................(87)}$$

Note.—Since there are any number of angles whose tangent has a given value x, and these are included in the formula $n.180°+A$, it follows that the formulae (85, 86) only hold when the inverse tangents are properly chosen. The more general form of (85) in radians is

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} + n\pi \quad \text{.........(85A)}$$

and a similar modification is required in the other cases.

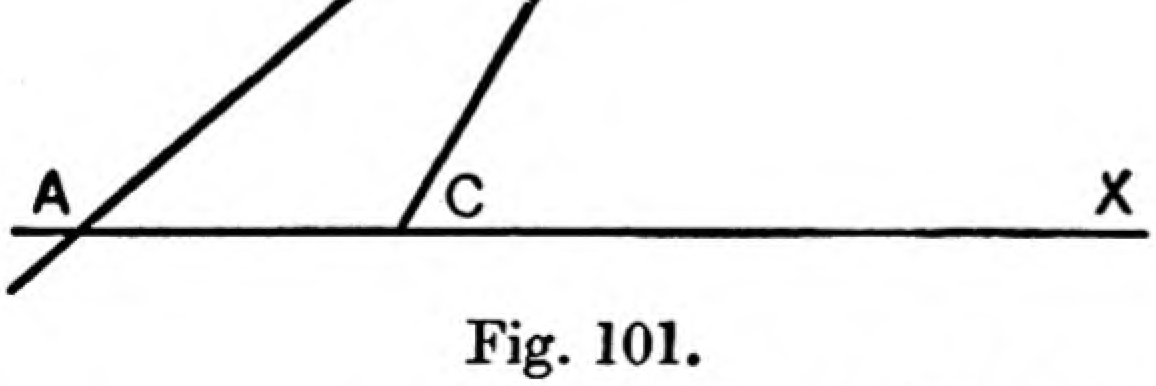


Fig. 101.

Ex. 1. Two lines AP and CP make angles $\tan^{-1} m_1$, $\tan^{-1} m_2$ with another line AX. Find the angle they make with one another.

$\angle PAX = \tan^{-1} m_1$;

$\therefore \quad \tan PAX = m_1$;

$\angle PCX = \tan^{-1} m_2$;

$\therefore \quad \tan PCX = m_2$;

$\angle APC = \angle PCX - \angle PAX$;

$$\therefore \quad \tan APC = \frac{\tan PCX - \tan PAX}{1+\tan PCX \tan PAX} = \frac{m_2 - m_1}{1+m_1 m_2}.$$

$$\therefore \quad \angle APC = \tan^{-1} \frac{m_2 - m_1}{1+m_1 m_2}.$$

Ex. 2. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{1}{4}\pi$.

For $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}\dfrac{\frac{5}{6}}{1-\frac{1}{6}}=\tan^{-1}1.$

Also, the *principal values* of $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$ are acute angles less than $\frac{1}{4}\pi$; hence their sum must be between 0 and π, and the only such angle whose tangent is 1 is $\frac{1}{4}\pi$; $\therefore\ \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{1}{4}\pi$.

Ex. 3. To find the value of $\tan^{-1}2+\tan^{-1}3$.

$$\tan^{-1}2+\tan^{-1}3=\tan^{-1}\frac{2+3}{1-2.3}=\tan^{-1}\frac{5}{-5}$$
$$=\tan^{-1}(-1)=-45^\circ \text{ or } 135^\circ\text{, etc.}$$

But, if the *principal values* of $\tan^{-1}2$ and $\tan^{-1}3$ are taken, these are the numerically least angles whose tangents are 2 and 3, and they lie between 0 and 90°. Therefore $\tan^{-1}2+\tan^{-1}3$ lies between 0 and 180°.

Hence 135° or $\frac{3}{4}\pi$ is the only admissible value in this case.

Ex. 4. Prove that $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{1}{4}\pi$.

Let $\tan^{-1}\frac{1}{3}=\alpha,\quad \tan^{-1}\frac{1}{5}=\beta,\quad \tan^{-1}\frac{1}{7}=\gamma,\quad \tan^{-1}\frac{1}{8}=\delta;$

$$\therefore\quad \tan\alpha=\tfrac{1}{3},\quad \tan\beta=\tfrac{1}{5},\quad \tan\gamma=\tfrac{1}{7},\quad \tan\delta=\tfrac{1}{8};$$

$$\therefore\quad \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}}=\frac{8}{14}=\frac{4}{7},$$

$$\tan(\gamma+\delta)=\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}=\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}=\frac{15}{55}=\frac{3}{11};$$

therefore
$$\tan(\alpha+\beta+\gamma+\delta)$$
$$=\frac{\tan(\alpha+\beta)+\tan(\gamma+\delta)}{1-\tan(\alpha+\beta)\tan(\gamma+\delta)}=\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\cdot\frac{3}{11}}=\frac{65}{65}=1=\tan\frac{\pi}{4}$$

therefore
$$\alpha+\beta+\gamma+\delta=\frac{\pi}{4}.$$

Ex. 5. Find the tangent of $4\tan^{-1}\frac{1}{5}-\frac{1}{4}\pi$.

$$2\tan^{-1}\tfrac{1}{5}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}=\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}=\tan^{-1}\tfrac{5}{12};$$

$$\therefore\quad 4\tan^{-1}\tfrac{1}{5}=2\tan^{-1}\tfrac{5}{12}=\tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}}=\tan^{-1}\tfrac{120}{119};$$

$$\therefore\quad 4\tan^{-1}\tfrac{1}{5}-\frac{\pi}{4}=\tan^{-1}\tfrac{120}{119}-\tan^{-1}1=\tan^{-1}\frac{\frac{120}{119}-1}{1+\frac{120}{119}}$$
$$=\tan^{-1}\frac{120-119}{119+120}=\tan^{-1}\tfrac{1}{239};$$

$$\therefore\quad \text{required tangent}=\tfrac{1}{239}.$$

157. Multiples and submultiples of inverse functions.

The student will find it an instructive exercise to obtain the inverse formulae corresponding to the formulae for multiple and submultiple angles established in Chap. XII. The inverse formulae are collected opposite the corresponding direct formulae below; but it is much better, in the first place, to obtain the results independently, and afterwards compare them with the table. Even if this be not done, the student should establish the conclusions by putting the sine, cosine, or tangent of A equal to x as the case may be.

Direct Formulae.	Inverse Formulae.
$\sin 2A = 2 \sin A \cos A,$	$2 \sin^{-1} x = \sin^{-1}\{2x\sqrt{(1-x^2)}\},$
$\cos 2A = 1-2 \sin^2 A,$	$2 \sin^{-1} x = \cos^{-1}(1-2x^2),$
$\cos 2A = 2 \cos^2 A-1,$	$2 \cos^{-1} x = \cos^{-1}(2x^2-1),$
$\tan 2A = \dfrac{2 \tan A}{1-\tan^2 A}$	$2 \tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}.$
$\sin 3A = 3 \sin A-4 \sin^3 A,$	$3 \sin^{-1} x = \sin^{-1}(3x-4x^3),$
$\cos 3A = 4 \cos^3 A-3 \cos A,$	$3 \cos^{-1} x = \cos^{-1}(4x^3-3x),$
$\tan 3A = \dfrac{3 \tan A-\tan^3 A}{1-3 \tan^2 A}.$	$3 \tan^{-1} x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}.$
$\sin \tfrac{1}{2}A = \sqrt{\dfrac{1-\cos A}{2}},$	$\tfrac{1}{2} \cos^{-1} x = \sin^{-1}\sqrt{\dfrac{1-x}{2}},$
$\cos \tfrac{1}{2}A = \sqrt{\dfrac{1+\cos A}{2}}.$	$\tfrac{1}{2} \cos^{-1} x = \cos^{-1}\sqrt{\dfrac{1+x}{2}}.$

Caution.—*These inverse formulae are not to be regarded as known fundamental formulae,* and they must therefore *not* be assumed in the solution of problems. They are given here solely as illustrative examples to familiarise the student with the inverse notation, and not to be remembered.

EXAMPLES XIV.

1. Find $\sin 18°$ and $\sin 54°$, and show that they are the roots of the equation $4x^2-2x\sqrt{5}+1 = 0$.

Solve the following equations (2–65):—

2. $\sin 2x = \cos 3x$.

3. $\sin 3\theta = \sin 4\theta$.

4. $\tan\left(\dfrac{\pi}{2\sqrt{2}}\sin\theta\right) = \cot\left(\dfrac{\pi}{2\sqrt{2}}\cos\theta\right)$.

5. $\tan(2\pi\cos\theta) = \cot(2\pi\sin\theta)$.

6. $\tan(\pi\sin\theta) = \cot\left(\dfrac{\pi}{2}\cos\theta\right)$.

7. $\sin(\pi\cos x) = \cos(\pi\sin x)$.

8. $\sin(\theta+a) = \cos(\theta-a)$.

9. $\sin 10\theta - \sin 4\theta = \sin 3\theta$.
10. $\sin 3\theta + \sin 2\theta + \sin\theta = 0$.
11. $\cos\theta + \cos 3\theta + \cos 5\theta = 0$.
12. $\sin 2\theta + \sin 3\theta + \sin 4\theta = 0$.
13. $\cos 2\theta + \cos 3\theta + \cos 4\theta = 0$.
14. $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
15. $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$.
16. $\sin 7\theta - \sin 5\theta = \sin 3\theta - \sin\theta$.
17. $\cos\theta + \cos 3\theta = \cos 2\theta + \cos 4\theta$.
18. $\sin 4\theta = \sin\theta$.
19. $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$.
20. $\cot\theta + \tan\theta = 4$.
21. $\cos(a-\theta)\cos a = \cos\theta$.
22. $\sin 6\theta = 2\sin 4\theta - \sin 2\theta$.
23. $\sin 4\theta + \sin 6\theta = \cos\theta$.
24. $2\cos 2\theta - 2\sin\theta = 1$.
25. $\cos 3\theta + 2\cos\theta = 0$.
26. $\sin a + \sin(\theta - a) + \sin(2\theta + a) = \sin(\theta + a) + \sin(2\theta - a)$.
27. $\sqrt{2}\sin\theta - \cos\theta = \sqrt{2}$.
28. $\sin\theta + \cos\theta = 1$.
29. $\sin\theta - \cos\theta = 1$.
30. $\sin 2\theta + \sqrt{3}\cos 2\theta = 1$.
31. $\sin\theta + \cos a = \cos\theta + \sin a$.
32. $\cos\theta + \sqrt{3}\sin\theta = 1$.
33. $a\cos\theta + b\sin\theta = a\sin a + b\cos a$.
34. $\sec\theta - \tan\theta = \sec a + \tan a$.
35. $\cos\theta + 2\sin\theta = \sqrt{5}$.
36. $\cos\theta + \tan\theta = \sec\theta$.
37. $\sin\theta\cos\theta + \sin a\cos a = \sin(a+\theta)$.
38. $\cot^2\theta - \tan^2\theta = 2\operatorname{cosec}\theta\sec\theta$.
39. $3\tan^2\theta + 8\cos^2\theta = 7$.
40. $2\sin(a+\theta)\sin(a-\theta) = 1 + 2\sqrt{2}\cos a\sin\theta$.
41. $\cos 2\theta + \cos 2\beta - 2\sqrt{2}\cos\theta\cos\beta + 1 = 0$.
42. $\cot^2\theta\tan^4 a + 1 = \cos^2\theta\sec^4 a$.
43. $\cos^3\theta - \cos\theta\sin\theta + \sin^3\theta = 1$.
44. $\tan^2\theta = 3\operatorname{cosec}^2\theta - 1$.
45. $\sin 2\theta = 3\tan\theta\cos 2\theta$.
46. $\cos 4\theta + 3\sin 2\theta - 2 = 0$.
47. $2\sin^2 2\theta + \sin^2 4\theta = 2$.
48. $\cos 2\theta + \sin\theta + \cos^2\theta = \frac{7}{4}$.
49. $\sin^2\theta + \cos^2 2\theta = \frac{3}{4}$.
50. $\sec\theta\operatorname{cosec}\theta + 2\cot\theta = 4$.
51. $2\sin\theta\sin 3\theta = \sin^2 2\theta$.
52. $\sqrt{3}\tan^2\theta + 1 = (1+\sqrt{3})\tan\theta$.
53. $\sin(3\theta + a)\cos(3\theta - a) = \cos^2\left(\frac{\pi}{4} - a\right)$.
54. $1 + \sin^2\theta = 3\sin\theta\cos\theta$.
55. $\tan 3\theta = (3 + 2\sqrt{2})\tan\theta$.
56. $\cos 2\theta - \sin\theta = \frac{1}{2}$.
57. $\cos 3\theta - (\sqrt{3}+1)\cos 2\theta + (\sqrt{3}+3)\cos\theta = \sqrt{3}+1$.
58. $2(1+\tan\theta) = (1-\tan\theta)\sec 2\theta$.
59. $\sin 2\phi - 2\tan\phi + \frac{1}{2\sqrt{3}} = 0$.
60. $\tan\theta + \tan 3\theta = 2\tan 2\theta$.

61. $\sin 3\theta = 4+(3-8\sqrt{\sin\theta})\sin\theta$.

62. $\dfrac{\sin\theta}{\cos\theta-\cos\alpha}+\dfrac{\cos\theta}{\sin\alpha-\sin\theta}=\dfrac{1}{\sin(\alpha-\theta)}$.

63. $\operatorname{cosec}^4\theta-\sec^4\theta = 2\operatorname{cosec}^3\theta\sec^3\theta$.

64. $4\sec 2\theta\operatorname{cosec}2\theta-\tan^3 2\theta = \cot 2\theta+2\tan 2\theta$.

65. $\dfrac{\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}}{\sqrt{1+\sin\alpha}-\sqrt{1-\sin\alpha}}=\frac{1}{2}\left(\tan\dfrac{\theta}{2}+\tan\dfrac{\alpha}{2}\right)\sqrt{\dfrac{\sin\theta}{\sin\alpha}}$.

66. Find the value of $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{12}$.

67. Prove that $\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{2}=n\pi+\dfrac{\pi}{4}$.

68. If $\sin^{-1}m+\sin^{-1}n=\dfrac{\pi}{2}$, show that $\sin^{-1}m=\cos^{-1}n$, and that $m\sqrt{1-n^2}+n\sqrt{1-m^2}=1$.

69. If $\alpha=\tan^{-1}\frac{1}{7}$, $\beta=\tan^{-1}\frac{1}{3}$, show that $\cos 2\alpha=\sin 4\beta$.

70. If $y=\tan^{-1}\dfrac{\sqrt{1+n^2}-\sqrt{1-n^2}}{\sqrt{1+n^2}+\sqrt{1-n^2}}$, show that $n^2=\sin 2y$.

71. Prove that

$$2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}\right).$$

72. If $n=\cot^{-1}\sqrt{\cos\alpha}-\tan^{-1}\sqrt{\cos\alpha}$, prove that

$$\sin n=\tan^2\frac{\alpha}{2}.$$

73. Find the value of $\cos 4(\tan^{-1}a)$ in terms of a.

74. If $\cos^{-1}\dfrac{x}{a}+\cos^{-1}\dfrac{y}{b}=\alpha$, prove that

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

PROVE the following relations (75–113):—

75. $\tan^{-1}\frac{4}{3}+\tan^{-1}7=135°$.

76. $\tan^{-1}1+\tan^{-1}(2-\sqrt{3})=60°$.

77. $\tan^{-1}\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\dfrac{3\pi}{4}-\tan^{-1}\sqrt{\dfrac{3}{2}}$.

78. $\cot^{-1}\frac{7}{5}+\cot^{-1}6=45°$.

79. $\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}=45°$.

80. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=45°$.

81. $\tan^{-1}\dfrac{m}{m+1}+\tan^{-1}\dfrac{1}{2m+1}=\dfrac{\pi}{4}$.

82. $\tan^{-1}\dfrac{m-1}{m}+\tan^{-1}\dfrac{1}{2m-1}=\dfrac{\pi}{4}$.

83. $\cot^{-1}\dfrac{1}{3}-\cot^{-1}3=\cot^{-1}\dfrac{3}{4}$.

84. $\sin^{-1}\dfrac{2mn}{m^2+n^2}+\sin^{-1}\dfrac{m^2-n^2}{m^2+n^2}=\dfrac{\pi}{2}$.

85. $\tan^{-1}\dfrac{x-\sqrt{x^2-4}}{2\sqrt{x+1}}+\tan^{-1}\dfrac{x+\sqrt{x^2-4}}{2\sqrt{x+1}}=\tan^{-1}\sqrt{x+1}.$

86. $\cos^{-1}\sqrt{\dfrac{2}{3}}-\cos^{-1}\dfrac{\sqrt{6}+1}{2\sqrt{3}}=\dfrac{\pi}{6}.$

87. $\tan^{-1}\frac{3}{4}=\frac{1}{2}\tan^{-1}\frac{24}{7}=\frac{1}{3}\cos^{-1}(-\frac{44}{125}).$

88. $\tan^{-1}\frac{4}{3}=\frac{1}{2}\tan^{-1}(-\frac{24}{7})=\frac{1}{3}\cos^{-1}(-\frac{117}{125}).$

89. $2\tan^{-1}\frac{2}{3}-\operatorname{cosec}^{-1}\frac{5}{3}=\sin^{-1}\frac{33}{65}.$

90. $\cot^{-1}3+\operatorname{cosec}^{-1}\sqrt{5}=\dfrac{\pi}{4}.$ 91. $\tan^{-1}\dfrac{2}{3}+\tan^{-1}\dfrac{3}{4}-\tan^{-1}\dfrac{11}{23}=\dfrac{\pi}{4}$

92. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{7}{9}+\tan^{-1}\frac{1}{7}=0.$

93. $\tan^{-1}\frac{1}{2}+\operatorname{cosec}^{-1}\sqrt{10}=45^\circ.$

94. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{5}=45^\circ-\tan^{-1}\frac{1}{47}.$

95. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}+\cos^{-1}\frac{56}{65}=90^\circ.$

96. $\cos^{-1}\dfrac{1}{2}+\sin^{-1}\dfrac{1}{2}+\cos^{-1}\dfrac{\sqrt{3}}{2}=120^\circ.$

97. $\tan^{-1}m+\tan^{-1}n=\cos^{-1}\dfrac{1-mn}{\sqrt{1+m^2}\sqrt{1+n^2}}.$

98. $\tan^{-1}\dfrac{1}{4}+\tan^{-1}\dfrac{2}{9}+\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{8}=\dfrac{\pi}{4}.$

99. $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{3}{5}=90.$ 100. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}.$

101. $\tan^{-1}\dfrac{1}{7}+\tan^{-1}\dfrac{1}{5}+\cot^{-1}\frac{23}{11}=\dfrac{\pi}{4}.$

102. $\cot^{-1}\dfrac{ab+1}{a-b}+\cot^{-1}\dfrac{bc+1}{b-c}+\cot^{-1}\dfrac{ca+1}{c-a}=0.$

103. $\cos^{-1}x=\sin^{-1}\sqrt{\dfrac{1-x}{2}}+\cos^{-1}\sqrt{\dfrac{1+x}{2}}.$

104. $\cos^{-1}\frac{63}{65}+\tan^{-1}\frac{1}{5}+\cot^{-1}5=\sin^{-1}\frac{3}{5}.$

105. $\tan^{-1}\dfrac{x\cos\phi}{1-x\sin\phi}-\tan^{-1}\dfrac{x-\sin\phi}{\cos\phi}=\phi.$

106. $3\tan^{-1}a=\tan^{-1}\dfrac{3a-a^3}{1-3a^2}.$

107. $\tan^{-1}x+\tan^{-1}y+\tan^{-1}\dfrac{1-x-y-xy}{1+x+y+xy}=\dfrac{\pi}{4}.$

108. $\dfrac{1}{2}\sin^{-1}\dfrac{4}{5}+\tan^{-1}\dfrac{1}{3}=\dfrac{\pi}{4}$

109. $\sin^{-1}x-\sin^{-1}y=\cos^{-1}(\sqrt{1-x^2-y^2+x^2y^2}+xy).$

110. $\tan^{-1}(\cot A)-\tan^{-1}(\tan A) = n\pi + \frac{\pi}{2} - 2A.$

111. $2\cot^{-1}x = \operatorname{cosec}^{-1}\frac{1+x^2}{2x}.$

112. $2\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{24}{25} = \tan^{-1}\frac{24}{7}.$

113. $2\sin^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25} = \cos^{-1}\frac{7}{25}.$

Solve the equations (114–126):—

114. $\tan^{-1}(x+1)\sqrt{2}-\tan^{-1}\frac{x-1}{\sqrt{2}} = \cot^{-1}4\sqrt{2}.$

115. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x).$

116. $\sin(\cot^{-1}\frac{1}{2}) = \tan(\cos^{-1}\sqrt{x}).$

117. $\tan^{-1}x+\tan^{-1}(1-x) = 2\tan^{-1}\sqrt{x-x^2}.$

118. $\sin^{-1}x+\sin^{-1}(1-x) = \cos^{-1}x.$

119. $\sec^{-1}a-\sec^{-1}b = \sec^{-1}\frac{x}{b}-\sec^{-1}\frac{x}{a}.$

120. $\cos^{-1}\frac{1-x^2}{1+x^2}+\tan^{-1}\frac{2x}{1-x^2} = \frac{4\pi}{3}.$

121. $\tan^{-1}(1-x)+\tan^{-1}(1+x) = \frac{\pi}{4}.$

122. $3\tan^{-1}(x+1)-\tan^{-1}\frac{x}{2-x} = 2\tan^{-1}(x-1).$

123. $\sin^{-1}\frac{x}{a}+\cos^{-1}\frac{x}{b} = \frac{\pi}{6}.$

124. $\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c} = \frac{\pi}{2}.$

125. $\tan^{-1}x+\tan^{-1}2x+\tan^{-1}3x = \pi.$

126. $\tan^{-1}(x+1)+\cot^{-1}(x-1) = \sin^{-1}\frac{4}{5}+\cos^{-1}\frac{3}{5}.$

CHAPTER XV.

LOGARITHMS.

158. As logarithms play an important part in simplifying trigonometrical calculations, especially in the solution of triangles, it will be convenient to devote a chapter to their more important properties before proceeding further.

Suppose two (variable) quantities x, y to be connected by the relation $y = 10^x$. If we assign to the index x any series of values in arithmetical progression, it is easy to see that the corresponding values of y will be in geometrical progression, for instance, taking integral values of x, we have the series—

[Indices]	$x = \ldots,$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3,$	$4, \ldots;$
[Powers of ten]	$y = \ldots,$	$10^{-2},$	$10^{-1},$	$10^0,$	$10^1,$	$10^2,$	$10^3,$	$10^4;$
	$=$	$\cdot 01,$	$\cdot 1,$	$1,$	$10,$	$100,$	$1{,}000,$	$10{,}000,$

where the members of the second or third row are in geometrical progression with 10 for their common ratio.

If we assign intermediate fractional values to x, the Theory of Indices in Algebra enables us to obtain values for 10^x or y.

Thus, giving values in arithmetic progression to x $(0, \cdot 1, \cdot 2, \cdot 3, \ldots, \cdot 9, 1)$, the corresponding values of y will form a geometric progression whose first term is 1 and whose last term is 10, and whose intermediate terms are therefore the *nine geometric means* between 1 and 10 (*not* the integers $1, 2, 3, \ldots, 9$).

Moreover, 10^x increases as x increases, and by taking x positive and sufficiently great, we may make 10^x as large as we please; while, by taking x negative and sufficiently great, 10^x may be made as small as we please.

Conversely, if, instead of x being given, we assign a given positive value to y, then a quantity x must exist which is connected with y by the relation $10^x = y$. This quantity is called the *logarithm* of y to the base 10, and the fact is stated thus—

$$x = \log_{10} y.$$

We might have started with the powers of any other number instead of 10, and that number would be called the base of our logarithms.

Thus, if $M = a^n$, then n is called the logarithm of M to the base a and is written thus—

$$n = \log_a M.$$

Hence the following definition:—

Def.—The **logarithm of a number to a given base** is the *index* of the power to which the base must be raised to produce that number.

Ex. Thus

$$\log_3 9 = 2, \qquad \text{since } 9 = 3^2;$$
$$\log_2 8 = 3, \qquad \text{since } 8 = 2^3;$$
$$\log_5 \tfrac{1}{25} = -2, \qquad \text{since } \tfrac{1}{25} = 5^{-2}.$$

The definition of a logarithm is readily exhibited by the identities

$$\log_a (a^n) = n, \quad a^{\log_a x} = x \quad \ldots\ldots\ldots\ldots\ldots\ldots (88)$$

159. **Properties of logarithms derived from those of indices.** Since a logarithm is an index, and the two relations $a^x = y$ and $x = \log_a y$ are equivalent, it follows that the properties of logarithms can be derived from the results proved in the Theory of Indices in Algebra.

The following is a list of the principal properties of logarithms placed opposite the properties of indices from which they may be derived.

	Properties of Indices.	Properties of Logarithms.	
(i)	$a^0 = 1,$	$\log_a 1 = 0$	(89)
(ii)	$a^1 = a,$	$\log_a a = 1$	(90)
(iii)	$a^{\frac{1}{2}} = \sqrt{a},$	$\log_a (\sqrt{a}) = \frac{1}{2}$	(91)
(iv)	$a^{-1} = 1/a,$	$\log_a (1/a) = -1$	(92)
(v)	$a^{\infty} = \infty$ (if $a>1$),	$\log_a \infty = \infty$ (if $a>1$)	(93)
(vi)	$a^{-\infty} = 0$ (if $a>1$),	$\log_a 0 = -\infty$ (if $a>1$)	(94)
(vii)	$a^x a^y = a^{x+y},$	$\log_a MN = \log_a M + \log_a N$	(95)
(viii)	$a^x/a^y = a^{x-y},$	$\log_a M/N = \log_a M - \log_a N$	(96)
(ix)	$(a^x)^n = a^{nx},$	$\log_a M^n = n \log_a M$	(97)
(x)	$\sqrt[n]{a^x} = a^{x/n},$	$\log_a \sqrt[n]{M} = (\log_a M)/n$	(98)

The first six deductions are obvious, the last four readily follow on putting $a^x = M$, $a^y = N$; and $\therefore\ x = \log_a M$, $y = \log_a N$, as we shall now prove.

160. To prove that

$$\log_a MN = \log_a M + \log_a N \text{ ....(95)}$$

Put $\log_a M = x$, $\log_a N = y$;

$$\therefore\ M = a^x,\ N = a^y;$$

$$\therefore\ MN = a^{x+y},$$

$$\therefore\ \log_a MN = x+y \text{ (by def.)} = \log_a M + \log_a N.$$

In the same way, it may be proved that

$$\log_a MNP\ldots = \log_a M + \log_a N + \log_a P + \text{ ......(95}a\text{)}$$

161. To prove that

$$\log_a \frac{M}{N} = \log_a M - \log_a N \text{ .... (96)}$$

Put $\log_a M = x$, $\log_a N = y$; $\therefore\ M = a^x$, $N = a^y$;

$$\therefore\ \frac{M}{N} = a^{x-y};$$

$$\therefore\ \log_a \frac{M}{N} = x - y = \log_a M - \log_a N.$$

162. To prove that

$$\log_a M^n = n \log_a M \text{ ....(97)}$$

Put $\log_a M = x$; $\therefore\ M = a^x$; $\therefore\ M^n = a^{nx}$;

$$\therefore\ \log_a M^n = nx = n \log_a M.$$

163. To prove that

$$\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M \text{ ....(98)}$$

Put $\log_a \sqrt[n]{M} = x$; $\therefore\ \sqrt[n]{M} = a^x$; $\therefore\ M = a^{nx}$;

$$\therefore\ \log_a M = nx = n \log_a \sqrt[n]{M};$$

$$\therefore\ \log_a \sqrt[n]{M} = \log_a M \div n.$$

164. **Summary.**—The preceding four results may also be stated in words thus:

The logarithm of a product *is equal to the sum of the logarithms of its factors* .... **(95)**

The logarithm of a quotient *is equal to the algebraic difference of the logarithms of the dividend and the divisor* .... **(96)**

The logarithm of any power *of a number is equal to the product of the logarithm of the number into the index of the power* .... **(97)**

The logarithm of any root *of a number is equal to the logarithm of the number divided by the index of the root* .... **(98)**

It hence follows that, by working with logarithms of numbers instead of the numbers themselves, the process of

Multiplication is replaced by		addition,
Division	,,	subtraction,
Involution	,,	multiplication,
Evolution	,,	division.

165. **Adaptation of formulae to logarithmic computation.**—Logarithms effect a great saving of labour in many numerical calculations, especially in trigonometry, where the process is further shortened by the extensive use of tables giving the *logarithms* of sines, cosines, etc., instead of the functions themselves. Their only compensating disadvantage is that they cannot be conveniently used in formulae involving the operations of *addition* and *subtraction*, for there is no very simple way of finding the logarithm of the sum or *difference* of two numbers whose logarithms are known. It is therefore important in practical work to use formulae which are adapted to logarithmic computation.

Ex. 1. Thus, if we have to find $\log c$ from the equation $c^2 = a^2 - b^2$ where a, b are given, we do not find the logarithms of a and b first, but we write the right-hand side as a product, thus $c^2 = (a-b)(a+b)$: and then, on taking logarithms of both sides, we have

$$2 \log c = \log (a-b) + \log (a+b),$$

whence $\log c$ can readily be found, $\log (a-b)$ and $\log (a+b)$ being obtained from the tables.

Ex. 2. Again, to find log (cos A − cos B), we transform the difference into a product (by Chap. XIII.), and obtain

$$\begin{aligned}\log(\cos A-\cos B) &= \log\{2\sin\tfrac{1}{2}(B+A)\sin\tfrac{1}{2}(B-A)\}\\ &= \log 2+\log\sin\tfrac{1}{2}(B+A)+\log\sin\tfrac{1}{2}(B-A).\end{aligned}$$

166. **Common logarithms.**—The *base* to which logarithms are commonly referred is **10**, the radix of the system of notation in common use. Such logarithms are called **Briggsian** or **Common Logarithms** on account of their having been introduced by Henry Briggs in the sixteenth century.

In common logarithms it is not usual to specify the base; thus $\log x$ means $\log_{10} x$.

One advantage of taking 10 as base is that the logarithm of any power of 10 can be at once written down by inspection.

Ex. Thus $\log_{10} 10{,}000 = 4$, since $10{,}000 = 10^4$;
$\log_{10} \cdot 001 = -3$, since $\cdot 001 = 10^{-3}$;
and so on.

Tables have been constructed giving the approximate values of the logarithms to base 10 of all numbers of not more than 5 digits, calculated to seven places of decimals. These are known as tables of seven-figure logarithms. For most practical calculations, however, **tables of five-figure logarithms** (*i.e.* logarithms calculated only to five places) are sufficiently accurate.

The logarithms of commensurable numbers other than powers of 10 are incommensurable. Like the quantity π, they cannot be represented by terminating or recurring decimals, but their values can be determined from theoretical considerations to any required degree of approximation.

167. **Characteristic and mantissa.** — In dealing with logarithms some of which are positive and others negative, it is found convenient in all cases to write them so that the decimal part is positive, the integral part being positive or negative according as the logarithm itself is positive or negative.

Def.—The positive fractional part is called the **mantissa**, and the integral part obtained after expressing the mantissa positively is called the **characteristic** of the logarithm.*

* In some books of tables the characteristic is called the **index** of the logarithm, although, strictly speaking, the whole logarithm is an index, and has been defined above as such.

Ex.—If the logarithm be $-2{\cdot}69897$, this

$$=-3+1-{\cdot}69897=-3+{\cdot}30103.$$

This is written $\bar{3}{\cdot}30103$; $\bar{3}$ or -3 is the *characteristic*, and $\cdot30103$ is the *mantissa*. It will be noted that the negative sign is placed *over* the characteristic instead of before it, to show that it applies only to the integral, and not to the decimal, portion.

DEF.—The **arithmetical complement** (A.C.) of a proper fraction is its defect from unity; *i.e.* the arithmetical complement of x is $1-x$.

This arithmetical complement is most readily found by subtracting the last significant figure of the decimal from 10, and the other figures before it from 9.

In converting a logarithm which is wholly negative into one with a positive mantissa, the simplest rule is to *replace the negative decimal part by its arithmetical complement (taken positive) for the mantissa, and add* -1 *to the integral part for the characteristic.*

Thus in the above example the mantissa $\cdot30103$ is the arithmetical complement of $\cdot69897$, and the characteristic -3 is found by adding -1 to -2.

*** *The next five articles* (§§ 168–172) *refer exclusively to common logarithms*, i.e. *logarithms to base* 10.

168. **The rule for the characteristic.**—The characteristic of the *common* logarithm of any number may always be found by inspection. We may conveniently introduce the subject by the following examples:—

Ex. 1. To find the characteristic of log 2351.

Here 2351 lies between 1,000 and 10,000, *i.e.* between 10^3 and 10^4;

$\therefore$ log 2351 lies between log 10^3 and log 10^4, *i.e.* between 3 and 4.

Hence log 2351 = 3 + proper fraction;

$\therefore$ the required characteristic is 3.

Ex. 2. To find the characteristic of log $\cdot0002351$.

Here $\cdot0002351$ lies between $\cdot0001$ and $\cdot001$, *i.e.* 10^{-4} and 10^{-3};

$\therefore$ log $\cdot0002351$ lies between log 10^{-4} and log 10^{-3}, *i.e.* between -4 and -3.

Hence log $\cdot0002351 = -4 +$ a positive proper fraction;

$\therefore$ the required characteristic is -4.

The characteristic of a logarithm to base 10 always depends on the position of the **first significant digit** of the number, *i.e.* the figure furthest to the left, other than a zero. We may now state and prove the following **rule for the characteristic**:—

If the first figure is in the unit's place, the characteristic is 0. **Add** 1 *for each place that the first figure is to the* **left** *of the unit's place,* **subtract** 1 *for each place that the first figure is to its* **right.**

[In Ex. 1 above, the first figure of 2351 (viz. 2) is 3 places to the *left* of the units' place; $\therefore$ characteristic $= 3$.

In Ex. 2, the first figure of ·0002351 is 4 places to *right* of the unit's place; $\therefore$ characteristic $= -4$, agreeing with the results already found.]

169. Proof of the rule.—CASE I. If the first digit is in the unit's place, the number lies between 1 and 10; hence its logarithm lies between $\log_{10} 1$ and $\log_{10} 10$, *i.e.* between 0 and 1, and is therefore a positive fraction. Hence its integral part, the characteristic, is zero.

CASE II. If the first digit is l places to the left of the unit's place, the number lies between 10^l and 10^{l+1}; hence its logarithm lies between l and $l+1$, and is equal to $l+$ a proper fraction. Thus the characteristic is l.

CASE III. If the first significant digit is after the decimal point r places to the *right* of the unit's place, the number lies between 10^{-r} and 10^{-r+1}.

Hence its logarithm lies between $-r$ and $-r+1$, and is equal to $-r+$a proper fraction. Since the decimal part must be positive, the characteristic is $-r$.

170. The rule for the mantissa.—If we know the logarithm of any number, we can at once write down the logarithm of that number multiplied or divided by any power of 10.

Ex. Having given $\log 525 = 2{\cdot}72016$, find $\log 52500$, $\log 5{\cdot}25$, and $\log {\cdot}00525$.

Here $52500 = 525\times 10^2$, $5{\cdot}25 = 525\div 10^2$, ${\cdot}00525 = 525\div 10^5$;

$$\begin{aligned} \therefore\quad \log 52500 &= \log 525+\log_{10} 10^2 = 2{\cdot}72016+2 = 4{\cdot}72016,\\ \log 5{\cdot}25 &= \log 525-\log_{10} 10^2 = 2{\cdot}72016-2 = 0{\cdot}72016,\\ \log{\cdot}00525 &= \log 525-\log_{10} 10^5 = 2{\cdot}72016-5 = \bar{3}{\cdot}72016. \end{aligned}$$

It will be noticed that the logarithms in the above example differ only in their integral parts, and not in their mantissae. This property may be stated generally in the following **rule for the mantissa**:—

Two numbers having the same significant figures have the same mantissa to their logarithms and differ only in the characteristics.

171. **Proof of the rule.**—Let M be any number having the same significant figures as N; then

$$M = N \times \text{some integral power of } 10 = N.10^n, \text{ suppose;}$$

therefore,
$$\log_{10} M = \log_{10} N + n \log_{10} 10$$
$$= \log_{10} N + n,$$

that is, the logarithms of N and $N.10^n$ to the base 10 differ by an integer n, which may be positive or negative. They therefore have the same mantissa, since, as in § 167, this mantissa is always made positive.

172. **Advantages of the base 10.**—From the rules for the characteristic and mantissa, we infer that the mantissa of the logarithm of any number to base 10 depends only on the sequence of digits in the number, and the characteristic only on the position of the first significant figure.

This property effects an enormous saving in the length of logarithmic tables. Tables of five-figure logarithms usually give the *mantissae* only (without the characteristics) for all numbers from 100 to 999, and these data suffice, with the aid of certain devices, to determine the *logarithm* of any number of not more than five significant digits (whatever be the position of these digits relative to the unit's place) the characteristic being found by inspection.

Ex. Given the mantissa of $\log 63225 = \cdot 80089$, to write down (i) the logarithms of 6·3225, 632·25, 632250, ·0063225; and (ii) the numbers whose logarithms are 3·80089 and $\bar{2}\cdot 80089$.

(i) By the rule for the mantissa, the mantissa of *each* logarithm is ·80089, and, by the rule for the characteristic, the four characteristics are 0, 2, 5, and -3.

Hence the four logarithms are 0·80089, 2·80089, 5·80089, and $\bar{3}\cdot 80089$.

(ii) Again, every number whose mantissa is ·80089 consists of the sequence of digits 63225. If the characteristic is 3, the first significant digit is 3 places to the left of the unit's place; if the characteristic is -2, it is 2 places to the right.

Hence 3·80089 and $\bar{2}\cdot 80089$ are the logarithms of 6322·5 and ·063225.

173. **Antilogarithms.**—The relation

$$y = \log_a x,$$

or $\quad y$ is the logarithm of x to base a,

can be expressed by saying that

$$x \text{ is the antilogarithm of } y \text{ to base } a,$$

$$\text{or} \quad x = \text{antilog}_a\, y.$$

This is, of course, only another way of saying that

$$x = a^y.$$

Ex.—log 3·69 = ·56703, so that antilog ·56703 = 3·69.

The position of the decimal point of the antilogarithm of a given number can thus be determined from a knowledge of the integral part of the given number by means of the following rule, which will be found to accord with the rule in § 168. The decimal part of the given number must first of all be made positive.

RULE.—*Place the decimal point immediately after the first figure of the antilogarithm. Then move the decimal point as many places to the* **right** *as there are* **positive** *units in the integral part,* i.e. *the characteristic, of the given number, or as many places to the* **left** *as there are* **negative** *units in the characteristic.*

Thus

$$\begin{aligned}
\text{antilog } \cdot56703 &= 3\cdot69 \\
\text{antilog } 1\cdot56703 &= 36\cdot9 \\
\text{antilog } 2\cdot56703 &= 369 \\
\text{antilog } 4\cdot56703 &= 36900 \\
\text{antilog } \bar{1}\cdot56703 &= \cdot369 \\
\text{antilog } \bar{3}\cdot56703 &= \cdot00369 \\
\text{antilog } \bar{4}\cdot56703 &= \cdot000369
\end{aligned}$$

By means of a table of antilogarithms, usually given in tables of five-figure logarithms, it is easy to find, to five significant figures, any number whose logarithm is given.

174. Transformation of bases of logarithms.—It is sometimes necessary to calculate logarithms referred to bases other than 10, and it is therefore convenient to be able to transform logarithms readily from one base to another.

Let N be the number, a and b the bases to which the logarithms of N are calculated. Then we shall prove that

$$\log_b N = \frac{\log_a N}{\log_a b} = N \log_a \log_b a \ldots\ldots\ldots\ldots (99)$$

Let $$\log_a N = x, \quad \log_a b = y;$$
$$\therefore \quad N = a^x, \quad b = a^y; \quad \therefore \quad a = b^{1/y};$$
$$\therefore \quad N = (b^{1/y})^x = b^{x/y}; \quad \therefore \quad \log_b N = \frac{x}{y} = \frac{\log_a N}{\log_a b}.$$
But $$a = b^{1/y};$$
$$\therefore \quad \frac{1}{y} = \log_b a; \qquad \therefore \quad \log_b N = \frac{x}{y} = \log_a N . \log_b a.$$

We have also proved incidentally that
$$\log_b a = \frac{1}{y} = \frac{1}{\log_a b},$$
whence $$\boldsymbol{\log_b a \times \log_a b = 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(100)}$$

Ex.—Find the logarithm of 3 to the base 2.
$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = \frac{47712}{30102} = 1{\cdot}5850 \ldots$$

175. Tabular logarithms of trigonometric functions.—As the sine and cosine of an angle are always less than unity, their logarithms are negative, and the same is true for the tangent of an angle less than 45°, or the cotangent of an angle greater than 45°. In such cases the introduction of negative characteristics is avoided by the use of what are called tabular logarithms.

DEF.—The **tabular logarithm** of any trigonometric function is the common logarithm of that function *increased by* **10.**

Tabular logarithms are denoted by the prefix L instead of log.

Thus $L \sin A = 10 + \log \sin A$, and this is read "tabular log sine A"; or, for brevity, "tabular sine A." Similarly, $L \tan A = 10 + \log \tan A$, $L \operatorname{cosec} A = 10 + \log \operatorname{cosec} A$, and so on, so that mentally we may remember that
$$\boldsymbol{L = 10 + \log \ldots}$$

For the sake of uniformity English trigonometric tables give the logarithms of *all* the functions of an angle increased by 10, even although this is unnecessary in the case of the secant and the cosecant. Since these are greater than unity, their ordinary logarithms are essentially positive.

Ex. 1. Given $\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$, find $L \sin 45°$, $L \tan 30°$, $L \operatorname{cosec} 60°$.

Since $$\sin 45° = \frac{1}{\sqrt{2}} = (2)^{-\frac{1}{2}},$$

$$\therefore \quad \log \sin 45° = -\tfrac{1}{2} \log 2 = -\cdot 15051 = \bar{1}\cdot 84949$$

and $$L \sin 45° = 10 + \log \sin 45° = 9\cdot 84949.$$

Since $$\tan 30° = \frac{1}{\sqrt{3}} = 3^{-\frac{1}{2}},$$

$$\therefore \quad \log \tan 30° = -\tfrac{1}{2} \log 3 = -\cdot 23856 = \bar{1}\cdot 76144$$

and $$L \tan 30° = 10 + \log \tan 30° = 9\cdot 76144.$$

Since $$\operatorname{cosec} 60° = \frac{2}{\sqrt{3}},$$

$$\therefore \quad \log \operatorname{cosec} 60° = \log 2 - \tfrac{1}{2} \log 3 = \cdot 30103 - \cdot 23856 = \cdot 06247$$

and $$L \operatorname{cosec} 60° = 10 + \log \operatorname{cosec} 60° = 10\cdot 06247.$$

Ex. 2. To express in tabular logarithmic notation the identities

$$\sin A = \frac{1}{\operatorname{cosec} A}, \quad \tan A = \frac{\sin A}{\cos A}, \quad \text{and} \quad \sin 2A = 2 \sin A \cos A.$$

Writing the first $\sin A \operatorname{cosec} A = 1$, and taking logarithms, we have
$$\log \sin A + \log \operatorname{cosec} A = 0;$$

$$\therefore \quad 10 + \log \sin A + 10 + \log \operatorname{cosec} A = 20,$$

or $$L \sin A + L \operatorname{cosec} A = 20.$$

The second identity gives

$$\log \tan A = \log \sin A - \log \cos A;$$

$$\begin{aligned} \therefore \quad 10 + \log \tan A &= 10 + \log \sin A - \log \cos A \\ &= 10 + \log \sin A - (10 + \log \cos A) + 10, \end{aligned}$$

or $$L \tan A = L \sin A - L \cos A + 10.$$

The third becomes

$$\log \sin 2A = \log 2 + \log \sin A + \log \cos A;$$

$$\therefore \quad 10 + \log \sin 2A = 10 + \log \sin A + 10 + \log \cos A - 10 + \log 2,$$

or $$\begin{aligned} L \sin 2A &= L \sin A + L \cos A - 10 + \log 2 \\ &= L \sin A + L \cos A + \overline{10}\cdot 30103. \end{aligned}$$

EXAMPLES XV.

[N.B.—In Examples 7 to 22, logarithms are calculated to the base 10, except where otherwise specified.

$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 47712.]$$

1. What is meant by "a system of logarithms"? What distinguishes one system from another?

2. Show that the sum of the logarithms of two numbers is equal to the logarithm of the product of the numbers.

3. Prove that the logarithm of any power of a number is the product of the logarithm of the number by the index of the power.

4. Prove that the logarithm of the quotient of two numbers is equal to the logarithm of the dividend diminished by the logarithm of the divisor.

5. Wherein lies the convenience of our logarithms being calculated to base 10? What is the value of $\log_{10} 10^{10}$?

6. Show that, in the common system of logarithms,

$$\log (N\times 10^n) = n+\log N.$$

Why would this not be true in any other system of logarithms? What would be the corresponding formula for logarithms to base a?

7. Find log of $\frac{1}{24}$, given log 2 and log 3.

8. Find log ·00625, given log 2.

9. Find log ·8 and log 4000, given log 2.

10. If $\log_{10} \frac{1025}{1024} = x$ and $\log_{10} 2 = y$, then $\log_{10} 41 = x+12y-2$.

11. Solve the equation $2^x = 5$, given $\log 2 = \cdot 30103$.

12. Given a system of logarithms to base a, how may the logarithms to base b be calculated?

13. What is the logarithm of $\sqrt{2}$ to base 2?

14. Calculate $\log_{\sqrt{10}} 100$ and $\log_{100} (\sqrt{10})$.

15. Given $\log_p x = \alpha$, $\log_q x = \beta$, prove that $\log_{\frac{p}{q}} x = \dfrac{\alpha\beta}{\beta-\alpha}$.

16. Prove that—

$$(a)\ (\log_a b)\times(\log_b a) = 1,$$

$$(b)\ \frac{\log_a\{\sqrt{(\log_a b)}\}}{\sqrt{(\log_a b)}} + \frac{\log_b\{\sqrt{(\log_b a)}\}}{\sqrt{(\log_b a)}} = 0.$$

17. Give a definition of the characteristic of a logarithm which will be applicable whether the number be less or greater than unity.

18. Deduce from the definition the characteristics of the logarithms of 3478·1, ·37481, 3·4781, ·00034781.

19. What are the characteristics of $\log_{10} \cdot 32572$ and $\log_{\sqrt{2}} 25$?

20. How many ciphers are there between the decimal point and the first significant figure in $(\frac{1}{3})^{100}$ when expressed as a decimal?

21. Prove that—

$$\text{antilog}_a\, x\times \text{antilog}_a\, y = \text{antilog}_a\, (x+y).$$

22. Prove that— $\quad \text{antilog}_a\, xy = (\text{antilog}_a\, x)^y$
$$= (\text{antilog}_a\, y)^x.$$

23. How many figures are there in the integral part of the antilogarithms (to base 10) of $\cdot146$, $2\cdot146$, $3\cdot146$?

24. Show that antilog $\bar{2}\cdot341$ lies between $\cdot1$ and $\cdot01$. How many zeros are there after the decimal point in the antilogarithm of $\bar{1}\cdot341$, $\bar{3}\cdot341$, $\bar{5}\cdot341$?

25. Between what limits is the tabular logarithmic sine of every angle contained ?

26. Given $\log 2 = \cdot30103$, $\log 3 = \cdot47712$, calculate $L \cos 45°$ and $L \tan 60°$.

	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
39	·59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55	66	77	88	99
40	·60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11	21	32	43	54	64	75	86	96
41	·61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	32	42	52	63	73	84	94
42	·62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51	61	71	82	92
43	·63347	63448	63548	63649	63749	63849	63948	64049	64147	64246	10	20	30	40	50	60	70	80	90
44	·64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10	20	29	39	49	59	68	78	88
45	·65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48	57	67	76	86
46	·66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47	56	65	74	84
47	·67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	36	46	55	64	73	82
48	·68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9	18	27	36	45	53	62	71	80
49	·69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	17	26	35	44	53	61	70	79
50	·69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43	52	60	69	77
51	·70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8	17	25	34	42	50	59	67	76
52	·71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	42	50	58	66	75
53	·72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41	49	57	65	73
54	·73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40	48	56	64	72
55	·74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	62	70
56	·74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	·75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38	45	53	60	68
58	·76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	44	52	59	67
59	·77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	·77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	58	65
61	·78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	36	43	50	57	64
62	·79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	42	49	56	63
63	·79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	20	27	34	41	48	54	61
64	·80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34	40	47	54	60

CHAPTER XVI.

ON THE USE OF TABLES.

176. A book of five-figure mathematical tables contains usually the following tables:—

(1) *Tables of logarithms* (or rather the mantissae of logarithms) of all numbers from 1 to 999.

(2) *Tables of antilogarithms* corresponding to all mantissae from ·000 to ·999.

(3) *Tables of tabular logarithms of the six trigonometric functions* calculated for angles at certain intervals from 0° to 90°. In Clive's Mathematical Tables the intervals are 6′. As explained in the last chapter, these tabular logarithms are the common logarithms increased by 10.

(4) *Tables of the trigonometric functions themselves* also calculated at the same intervals. These are called **natural** sines, cosines, etc., to distinguish them from the preceding, which are referred to as **logarithmic** sines, etc.

177. Tables of logarithms of numbers are used to find the logarithm of a given number.

A portion of a page of one of these tables is given on page 196.

178. Plan of the Table.—On examining the table we notice—

(1) That the column on the extreme left contains the numbers of two digits from 39 to 64.

The figures in this column correspond to the *first two* significant figures of the number whose logarithm is required.

(2) That we then have ten columns headed by the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively.

These figures correspond to the *third* significant figure of the number whose logarithm is required. (The dark line dividing these columns into two sets is inserted merely to assist the eye in reading.)

(3) That we also have a series of nine columns, headed by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and with the general heading, "Mean Differences."

These columns belong respectively to numbers whose *fourth* (and sometimes fifth) significant figures are 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively.

The student who uses Clive's Mathematical Tables will further notice that in the rows facing the numbers 10, 11, ... 23 there are certain single figure quantities with + or − prefixed. The use of these quantities will be explained in § 191. They may be disregarded in calculations in which we require a result correct to *three* significant figures only.

179. To find the logarithm of a number containing three significant figures.

Pick out the horizontal row corresponding to the first two significant figures and the vertical column corresponding to the third significant figure.

The five figures that are at the junction of this particular row and this particular column form the mantissa of the logarithm of the given number.

The decimal point is, for the sake of simplicity, omitted from all except the first column.

Thus, if log 524 is required, we look in the horizontal row facing 52 and in the vertical column with 4 at its head.

At the junction of these we find the figures 71933, so that the mantissa of the required logarithm is ·71933.

Similarly the mantissa of the logarithm of 639 is to be found in the row facing 63 and in the vertical column headed 9, and is therefore ·80550.

The characteristic of the logarithm should be determined by the rule in § 168.

Thus
$$\log 524 = 2{\cdot}71933,$$
$$\log {\cdot}0639 = \bar{2}{\cdot}80550.$$

Where the given number has only *one* or *two* significant figures we take it and put respectively *two* zeros or *one* after the significant figures and then proceed as above.

Thus the mantissa of the logarithm of 49 is the same as that of 490, and is therefore ·69020.

Similarly the mantissa of the logarithm of ·06 is the same as that of 600, and is therefore ·77815.

180. To find the logarithm of a number containing more than three significant figures.

For this purpose we must use the columns of "mean differences." The actual values of the quantities in these columns are not integers, but multiples of ·00001. They are written in the abbreviated form to save space.

Thus the 11 in the first mean difference column is really ·00011.

We therefore proceed as follows:—

Write down the mantissa corresponding to the logarithm of the first three significant figures.

Then pick out in the same row as the mantissa just taken the mean difference in the column with the fourth significant figure at its head. Add this quantity to the mantissa found.

Ex. To find log 50·64.

Mantissa of log 506		=	·70415
M.D. or mean difference for	4	=	34
∴ mantissa of log 5064		=	·70449
∴ (by § 168) log 50·64		=	1·70449

If the number whose logarithm is required contains *five* significant figures, the mean difference for the fifth figure is evidently one-tenth of what it would be if that figure were the fourth significant figure.

Ex. To find log ·0048097.

Mantissa of log 480			=	·68124
M.D. for	9		=	80
M.D. for		7	=	6 2
∴ mantissa of log 840	9	7	=	·68210 2
∴ (by § 168) log ·0048097			=	$\bar{3}$·68210

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Mean Differences. 1	2	3	4	5	6	7	8	9
·80	63096	63241	63387	63533	63680	63826	63973	64121	64269	64417	15	29	44	59	74	88	103	118	132
·81	64565	64714	64863	65013	65163	65313	65464	65615	65766	65917	15	30	45	60	75	90	105	120	135
·82	66069	66222	66374	66527	66681	66834	66988	67143	67298	67453	15	31	46	62	77	92	108	123	139
·83	67608	67764	67920	68077	68234	68391	68549	68707	68865	69024	16	32	47	63	79	95	110	126	142
·84	69183	69343	69503	69663	69823	69984	70146	70307	70469	70632	16	32	48	64	81	97	113	129	145
·85	70795	70958	71121	71285	71450	71614	71779	71945	72111	72277	17	33	50	66	83	99	116	132	149
·86	72444	72611	72778	72946	73114	73282	73451	73621	73790	73961	17	34	51	68	85	101	118	135	152
·87	74131	74302	74473	74645	74817	74989	75162	75336	75509	75683	17	35	52	69	87	104	121	138	156
·88	75858	76033	76208	76384	76560	76736	76913	77090	77268	77446	18	35	53	71	89	106	124	142	159
·89	77625	77804	77983	78163	78343	78524	78705	78886	79068	79250	18	36	54	72	91	109	127	145	163
·90	79433	79616	79799	79983	80168	80353	80538	80724	80910	81096	19	37	56	74	93	111	130	148	167
·91	81283	81470	81658	81846	82035	82224	82414	82604	82794	82985	19	38	57	76	95	113	132	151	170
·92	83176	83368	83560	83753	83946	84140	84333	84528	84723	84918	19	39	58	78	97	116	136	155	175
·93	85114	85310	85507	85704	85901	86099	86298	86497	86696	86896	20	40	60	79	99	119	139	158	178
·94	87096	87297	87498	87700	87902	88105	88308	88512	88716	88920	20	41	61	81	102	122	142	162	183
·95	89125	89331	89536	89743	89950	—·1	—·1	0	0	+·1	21	41	62	83	103	124	144	165	186
	0	—·1	+·1	+·1	+·1	90157	90365	90573	90782	90991	21	42	63	84	104	125	146	167	188
·96	91201	91411	91622	91833	92045	—·1	—·1	0	0	0	21	42	63	84	106	127	148	169	190
	—·1	0	0	+·1	+·1	92257	92470	92683	92897	93111	21	43	64	85	107	128	150	171	192
·97	93325	93541	93756	93972	94189	—·1	—·1	—·1	+·1	0	22	43	65	86	108	130	151	173	195
	0	—·1	0	+·1	+·1	94406	94624	94842	95060	95280	22	44	66	87	109	131	153	175	197
·98	59499	95719	95940	96161	96383	—·1	—·1	0	0	+·1	22	44	66	88	111	133	155	177	199
	—·1	0	0	+·1	+·1	96605	96828	97051	97275	97499	22	45	67	90	112	134	157	179	201
·99	97724	97949	98175	98401	98628	—·1	0	0	0	+·1	23	45	68	90	113	136	158	181	204
	—·1	0	0	+·1	0	98855	99083	99312	99541	99770	23	46	69	92	115	137	160	183	206

The logarithms cannot be relied upon for more than five decimal places in the mantissa; so that we omit the figure in the sixth place, adding 1 to the figure in the fifth place if the figure in the sixth place is more than 4.

In Clive's Mathematical Tables there are *two* sets of mean differences opposite each row of the numbers 10, 11, . . . 23 of the first column of Table I.

The upper row are the mean differences for mantissae of logarithms of numbers whose *third* significant figure is 0, 1, 2, 3, or 4. The lower row are the mean differences for mantissae of logarithms of numbers whose *third* significant figure is 5, 6, 7, 8, or 9.

These two sets are given in these cases because one set of mean differences for the fourth significant figure does not give sufficiently accurate results.

181. To find the antilogarithm of a given number, *i.e.* to find the number whose logarithm is given.

On page 200 we give a portion of a table of antilogarithms, taken from Clive's Mathematical Tables.

In obtaining the antilogarithm of a number we should note that—

(1) As in finding the logarithm of a number we disregard at first the position of the decimal point, and ultimately allow for its position when we assign the correct characteristic to the logarithm,

So in finding the antilogarithm of a given number we disregard at first the integral part (*i.e.* the *characteristic* part) of the number, and ultimately use that part to determine the correct position of the decimal point in the antilogarithm.

(2) As the tables give antilogarithms to *five* significant figures only, if the antilogarithm of the given number has more than five digits, the requisite number of zeros must be appended to the final number obtained. This number will, in any case, be correct only to five significant figures.

Ex. 1. To find antilog 2·8762, *i.e.* to find the number whose logarithm is 2·8762.

We first find antilog ·8762, *i.e.* the number whose logarithm is ·8762.

Antilog ·876 is found at the junction of the row commencing with ·87 and the column headed 6. The figures given are 75162.

Thus, disregarding for the present the position of the decimal place in the antilog, we get

$$\begin{aligned} \text{antilog } \cdot 876 &= 75162 \\ \text{M.D. for } \quad 2 &= \quad 35 \\ \therefore \quad \text{antilog } \cdot 8762 &= 75197 \\ \therefore \quad (\text{by § } 173) \qquad \text{antilog } 2{\cdot}8762 &= 751{\cdot}97 \end{aligned}$$

Ex. 2. To find antilog $\bar{2}{\cdot}97239$.

$$\begin{aligned} \text{Antilog } \cdot 972 \quad &= 93756 \\ 3 \quad &= \quad 65\,* \\ 9 \quad & \quad\; 19\,5 \\ \therefore \quad \text{antilog } \cdot 972\,3\,9 &= 93840\,5 \\ &= 93841 \\ \therefore \quad (\text{by § } 173)\ \text{antilog } \bar{2}{\cdot}97239 &= \cdot 093841 \end{aligned}$$

Ex. 3. To find $\sqrt{10}$ to four places of decimals.

$$\begin{aligned} \log \sqrt{10} &= \log\{(10)^{\frac{1}{2}}\} \\ &= \tfrac{1}{2} \log 10 \\ &= \tfrac{1}{2} \\ &= \cdot 5 \\ \therefore \quad \sqrt{10} &= \text{antilog } \cdot 5 \\ &= \text{antilog } \cdot 500 \\ &= 3{\cdot}1623. \end{aligned}$$

Ex. 4. To find, to 4 places of decimals, the cube root of 3.

$$\begin{aligned} \log \sqrt[3]{3} &= \log (3^{\frac{1}{3}}) \\ &= \tfrac{1}{3} \log 3 \\ &= \tfrac{1}{3} \times \cdot 47712 \\ &= \cdot 15904 \\ \therefore \quad \sqrt[3]{3} &= \text{antilog } \cdot 15904 \end{aligned}$$

$$\begin{aligned} \text{Now antilog } \cdot 159 \quad &= 14421 \\ \text{M.D. for } \quad 04 &= \quad 1\,3 \\ \therefore \quad \text{antilog } \cdot 15904 &= 14422 \\ \therefore \quad \sqrt[3]{3} &= 1{\cdot}4422 \end{aligned}$$

182. The Principle of Proportionate Differences.

The " Mean Differences " given in the tables are calculated by means of an important principle commonly referred to as the **Theory of Proportional Parts,** but sometimes more

* The two sets of mean differences are used as explained on page 201.

appropriately called the **Principle of Proportionate Differences.** *As applied to logarithms* this principle may be stated in the following terms:—

If a number be increased by a very small fraction of itself, the increase in the logarithm of the number is very approximately proportional to the increase in the number.

Hence, if N be any number, and h, k any two quantities, *both very small compared with* N, then, very approximately,

$$\log(N+h)-\log N : \log(N+k)-\log N = h : k.$$

Caution.—The principle does *not* assert that logarithms of numbers are proportional to the numbers; this is evidently not the case.

The Principle of Proportionate Differences is used in the calculation of the "Mean Differences" in the tables. The method of applying it to calculate results intermediate between those given in tables is known as "interpolation." See § 187.

183. Proof of the Principle of Proportionate Differences.

$$\frac{\log(N+h)-\log N}{\log(N+k)-\log N}=\frac{\log\dfrac{N+h}{N}}{\log\dfrac{N+k}{N}}$$

$$=\frac{\log\left(1+\dfrac{h}{N}\right)}{\log\left(1+\dfrac{k}{N}\right)}.$$

Now it is proved in Higher Algebra that

$$\log(1+x)=\mu x-\mu\frac{x^2}{2}+\mu\frac{x^3}{3}-\ldots$$

when $x<1$. $[\mu = \cdot 4343.]$

If, then, x is small as compared with unity,

$$\log(1+x)=\mu x \text{ approximately.}$$

But in the above h and k are small as compared with N, so that $\frac{h}{N}$ and $\frac{k}{N}$ are small as compared with unity.

$$\therefore\ \log\left(1+\frac{h}{N}\right)=\mu\frac{h}{N},$$

and $$\log\left(1+\frac{k}{N}\right)=\mu\frac{k}{N}, \text{ approximately.}$$

$$\therefore \quad \frac{\log(N+h)-\log N}{\log(N+k)-\log N}=\frac{\mu\dfrac{h}{N}}{\mu\dfrac{k}{N}}=\frac{h}{k},$$

which proves the proposition.

184. **Table of Tabular Logarithms of sine and cosine.**— The construction and use of these tables will easily be understood from the extract given opposite from the top and bottom of a page of actual tables.

Ex. 1. To find $L \sin 36° 42'$.

Pick out the horizontal row commencing with 36° and the vertical column headed 42′. The quantity at the junction is 77643.

$$\therefore \quad L \sin 36° 42' = 9{\cdot}77643.$$

When the number of minutes in the minute part of the angle is not a multiple of 6.

Take that angle, say A, next below the given angle, that is a multiple of 6′. Write down its L sin. Then from the same row pick out the "mean difference" corresponding to the excess of the given angle over the angle A. Add this to the value found for A.

Ex. 2. To find $L \sin 78° 40'$.

As in *Ex.* 1,
$$\begin{array}{lr} L \sin 78° 36' = & 9{\cdot}99135 \\ \text{M.D.} \quad 4' & 10 \\ \hline \therefore \quad L \sin 78° 40' = & 9{\cdot}99145 \end{array}$$

For a cosine, the following points should be noted:—

(*a*) The table is read from the bottom upwards.

(*b*) The *last* column is used for the number of degrees instead of the first column.

(*c*) The number of minutes in the angle must be read from the figures in the lowest row, and not from those in the highest row.

(*d*) Since the cosine of an angle less than 90° decreases as the angle increases, the "mean difference" must be subtracted, and not added.

LOGARITHMIC SINES.

Deg.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	60′	Mean Differences. 1′	2′	3′	4′	5′	Deg.
30	9·69897	70028	70159	70288	70418	70547	70675	70803	70931	71058	71184	21	43	64	86	107	59
31	9·71184	71310	71435	71560	71685	71809	71932	72055	72177	72299	72421	21	41	62	82	103	58
32	9·72421	72542	72663	72783	72902	73022	73140	73259	73377	73494	73611	20	40	60	79	99	57
33	9·73611	73727	73843	73959	74074	74189	74303	74417	74531	74644	74756	19	38	57	76	95	56
34	9·74756	74868	74980	75091	75202	75313	75421	75533	75642	75751	75859	18	37	55	74	92	55
35	9·75859	75976	76075	76182	76289	76395	76501	76607	76712	76817	76922	18	35	53	71	89	54
36	9·76922	77026	77130	77233	77336	77439	77541	77643	77744	77846	77946	17	34	51	68	85	53
37	9·77946	78047	78147	78246	78346	78445	78543	78642	78739	78837	78934	16	33	49	66	82	52
38	9·78934	79031	79128	79224	79319	79415	79510	79605	79699	79793	79887	16	32	48	64	79	51
39	9·79887	79981	80074	80166	80259	80351	80443	80534	80625	80716	80807	15	31	46	61	77	50
40	9·80807	80897	80987	81076	81166	81254	81343	81431	81519	81607	81694	15	30	44	59	74	49
41	9·81694	81781	81868	81955	82040	82126	82212	82297	82382	82467	82551	14	29	43	57	71	48
42	9·82551	82635	82719	82802	82885	82968	83051	83133	83215	83297	83378	14	28	41	55	69	47
43	9·83378	83459	83540	83621	83701	83781	83861	83940	84020	84098	84177	13	27	40	53	67	46
44	9·84177	84255	84336	84411	84489	84566	84643	84720	84796	84873	84948	13	26	39	51	64	45
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
75	9·98494	98515	98535	98555	98574	98594	98614	98633	98652	98671	98690	3	7	10	13	16	14
76	9·98690	98709	98728	98746	98765	98783	98801	98819	98837	98854	98872	3	6	9	12	15	13
77	9·98872	98890	98907	98924	98941	98958	98975	98991	99008	99024	99040	3	6	8	11	14	12
78	9·99040	99056	99072	99088	99104	99119	99135	99150	99165	99180	99195	3	5	8	10	13	11
79	9·99195	99209	99224	99238	99253	99267	99281	99294	99308	99322	99335	2	5	7	9	12	10
80	9·99335	99348	99362	99375	99388	99400	99413	99425	99438	99450	99462	2	4	6	8	11	9
81	9·99462	99474	99486	99497	99509	99520	99532	99543	99554	99565	99575	2	4	6	8	9	8
82	9·99575	99586	99596	99607	99617	99627	99637	99647	99657	99666	99675	2	3	5	7	8	7
83	9·99675	99684	99693	99702	99711	99720	99728	99737	99745	99753	99761	1	3	4	6	7	6
84	9·99761	99769	99777	99785	99792	99800	99807	99814	99821	99828	99834	1	2	4	5	6	5
85	9·99834	99841	99847	99854	99860	99866	99872	99878	99883	99889	99894	1	2	3	4	5	4
86	9·99894	99899	99904	99909	99914	99919	99923	99928	99932	99936	99940	1	2	2	3	4	3
87	9·99940	99944	99948	99952	99955	99959	99962	99965	99968	99971	99974	1	1	2	2	3	2
88	9·99974	99976	99979	99981	99983	99985	99987	99989	99990	99992	99993	0	1	1	1	2	1
89	9·99993	99995	99996	99997	99998	99998	99999	99999	10·000	10·000	10·000	0	0	0	0	1	0
	60′	54′	48′	42′	36′	30′	24′	18′	12′	6′	0′	1′	2′	3′	4′	5′	

LOGARITHMIC COSINES.

Ex. 3. To find $L\cos 9^\circ\, 20'$.

$$\begin{aligned} L\cos 9^\circ\, 18' &= 9\cdot 99425 \\ \text{M.D. for } 2' &= 4 \text{ Subtract} \\ \therefore\ L\cos 9^\circ\, 20' &= 9\cdot 99421. \end{aligned}$$

185. **Tables of Tabular Logarithms of trigonometric functions.**—The method of using the tables of the other trigonometric functions is similar to that explained in § 184.

The following points should be noted:—

(*a*) It will be seen that the integral part of the tabular logarithm is given in the $0'$ column only. The same figure applies to the other $6'$, $12'$, . . . columns, except where a black bar is placed over the figures, when the integer must be increased by unity.

E.g. in Clive's *Mathematical Tables*,

$$L\sin 5^\circ = 8\cdot 94030.$$

In the same row and in the column headed by $48'$ we find the figures $\overline{00456}$.

Thus $\qquad L\sin 5^\circ\, 48' = \mathbf{9}\cdot 00456.$

(*b*) As an angle increases from 0° to 90°, its

sine, tangent, and secant increase,

cosine, **co**tangent, and **co**secant decrease,

and the same is true for their tabular logarithms.

Therefore, when the tables are being used for the

$L\sin$, $L\tan$, and $L\sec$,

the mean differences must be *added*.

But when the tables are being used for the

$L\cos$, $L\cot$, and $L\operatorname{cosec}$,

the mean differences must be subtracted.

Ex. 1. Using the tables, find the value of $L\tan 84^\circ\, 23'$.

$$\begin{aligned} \text{From the tables} \quad L\tan 84^\circ\, 18' &= 11\cdot 00081 \\ \text{M.D. for } 5' &= 634 \\ \therefore\ L\tan 84^\circ\, 23' &= 11\cdot 00715. \end{aligned}$$

Ex. 2. Find the value of $L \cot 15° 19'$.

From the tables $L \cot 15° 18' = 10{\cdot}56293$

M.D. for $1' = 50$

The M.D. must be subtracted in the case of a cotangent,

$\therefore \quad L \cot 15° 19' = 10{\cdot}56243.$

186. **Principle of Proportionate Parts.**—We will now discuss the cases where no mean differences are given in the tables.

It has been seen that in certain cases *two* sets of mean differences were given for each row. This was done in order to obtain a greater degree of accuracy in the final result than would have been obtained with one set of mean differences only.

But in the cases in which no mean differences are given even two sets of mean differences would not be sufficient, and we have to find on each occasion the requisite mean difference by means of the Principle of Proportional Parts, which, when extended to the trigonometric functions, asserts that

If an angle be increased by a very small amount the change produced in the value of any function of the angle is approximately proportionate to the change in the angle.

When employed in connection with tables of logarithmic sines, tangents, and secants, the principle may be conveniently stated in the form of the equations

$$\frac{L \sin (A°+h) - L \sin A°}{L \sin (A°+k) - L \sin A°} = \frac{h}{k}$$

$$= \frac{L \operatorname{cosec} A° - L \operatorname{cosec} (A° + h)}{L \operatorname{cosec} A° - L \operatorname{cosec} (A° + k)},$$

$$\frac{L \tan (A°+h) - L \tan A°}{L \tan (A°+k) - L \tan A°} = \frac{h}{k}$$

$$= \frac{L \cot A° - L \cot (A° + h)}{L \cot A° - L \cot (A° + k)},$$

$$\frac{L \sec (A° + h) - L \sec A°}{L \sec (A° + k) - L \sec A°} = \frac{h}{k}$$

$$= \frac{L \cos A° - L \cos (A° + h)}{L \cos A° - L \cos (A° + k)}.$$

187. **Interpolation.**—The process by which the principle of proportionate parts is used is called **interpolation**, and is illustrated by the following examples:—

Ex. 1. To find $L \sin 3° 35'$.

From the tables
$$L \sin 3° 36' = 8{\cdot}79790$$
$$L \sin 3° 30' = 8{\cdot}78568$$
$$\therefore \text{ diff. for } 6' = {\cdot}01222$$

$\therefore$ by the principle of proportional parts
$$\text{diff. for } 5' = {\cdot}01222 \times \tfrac{5}{6}$$
$$= {\cdot}01018$$
$$\therefore L \sin 3° 35' = 8{\cdot}78568 + {\cdot}01018$$
$$= 8{\cdot}79586$$

Ex. 2. To find $L \cot 87° 44'$.

From the tables
$$L \cot 87° 48' = 8{\cdot}58451$$
$$L \cot 87° 42' = 8{\cdot}60383$$
$$\therefore \text{ diff. for } 6' = {\cdot}01932$$
$$\therefore \text{ diff. for } 2' = {\cdot}01932 \times \tfrac{2}{6}$$
$$= {\cdot}00644$$

Now the cotangent, and therefore the $L \cot$, decreases as the angle increases. Therefore we must *subtract* the difference from the values of $L \cot 87° 42'$.

Thus
$$L \cot 87° 44' = 8{\cdot}60383 - {\cdot}00644$$
$$= 8{\cdot}59739$$

188. **To find to the nearest minute the angle one of whose logarithmic functions is given.**

E.g. to find to the nearest minute the angle θ when the value of $L \sin \theta$ is given.

In the logarithmic sines table pick out from the columns headed 0′, 6′, 12′ ... the quantity nearest to, but not exceeding, the given value of $L \sin \theta$. Let the corresponding angle be A.

Subtract the value so found for $L \sin A$ from the given value. Let the difference be d.

Then find in the corresponding row which of the mean

differences is the nearest to the difference d. The number of minutes at the head of the column containing such mean difference should be added to A to give the required angle.

Ex. 1. Find to the nearest minute the angle θ such that

$$L \sin \theta = 9{\cdot}79.$$

From the table

$$\begin{aligned} L \sin 38^\circ &= 9{\cdot}78934 \\ L \sin \theta &= 9{\cdot}79000 \\ \hline \text{Diff.} \quad &= {\cdot}00066 \end{aligned}$$

The nearest mean difference in the same row is ·00064 corresponding to 4′ of angle.

$$\therefore \ \theta = 38^\circ + 4' = 38^\circ\, 4' \text{ to the nearest minute.}$$

189. The method is the same in the case of the L tan and L sec; but in the case of the L cos, L cot, and L cosec the number of minutes corresponding to the nearest mean difference must be *subtracted* instead of being added.

Ex. 2. To find to the nearest minute the angle whose L cot $= 10{\cdot}5$.
From the table $\qquad L \cot 17^\circ\, 36' = 10{\cdot}49864.$

The difference between 10·49864 and 10·5 = ·00136, which would be represented in the mean difference columns by 136.

The nearest mean difference to this in the same row as 10·49864 is 131, which gives a difference of 3′ in the angle.

Thus 10·5 is the L cotangent of $(17^\circ\, 36' - 3')$ or $17^\circ\, 33'$ to the nearest minute.

190. Where interpolation is required we proceed as in the following examples:—

Ex. 3. To find to the nearest minute the angle whose tangent is 6·587.

From the tables

$$\begin{aligned} \tan 81^\circ\, 24' &= 6{\cdot}61219 \\ \tan 81^\circ\, 18' &= 6{\cdot}53503 \\ \hline \therefore \text{ diff. for } \quad 6' &= {\cdot}07716. \end{aligned}$$

Let required angle be $\qquad 81^\circ\, 18' + x'$.

Then

$$\tan (81^\circ\, 18' + x') = 6{\cdot}58700$$

and

$$\begin{aligned} \tan 81^\circ\, 18' &= 6{\cdot}53503 \\ \hline \therefore \text{ diff. for } \quad x' &= {\cdot}05197 \end{aligned}$$

But

$$\text{diff. for } \quad 6' = {\cdot}07716.$$

Therefore, by the theory of proportional parts,

$$\frac{x'}{6'} = \frac{\cdot 05197}{\cdot 07716}.$$

$$\therefore\ x' = \frac{5197}{7716} \times 6' = 4{\cdot}2'.$$

Therefore to nearest minute $x' = 4'$,
and therefore required angle $= 81^\circ\ 18' + 4' = 81^\circ\ 22'$.

Ex. 4. To find to the nearest minute the angle whose L cosec is $11{\cdot}26$.

From the tables $L \operatorname{cosec} 3^\circ\ 12' = 11{\cdot}25320$
$L \operatorname{cosec} 3^\circ\ 6' = 11{\cdot}26697$

Diff. for $6' = \cdot 01377$.

Since the cosecant, and therefore the L cosec, of an angle decreases as the angle increases, the required angle is clearly *less* than $3^\circ\ 12'$. Let it be $(3^\circ\ 12' - x')$.

Then $L \operatorname{cosec} (3^\circ\ 12' - x') = 11{\cdot}26$
and $L \operatorname{cosec} 3^\circ\ 12' = 11{\cdot}25320$

$\therefore$ diff. for $x' = \cdot 00680$
and diff. for $6' = \cdot 01377$.

Therefore, by the theory of proportional parts,

$$\frac{x'}{6'} = \frac{680}{1377},$$

whence $x' = 2{\cdot}9' = 3'$ to nearest minute.

Therefore required angle
$= 3^\circ\ 12' - 3' = 3^\circ\ 9'$ to nearest minute.

191. **Corrections.**—Any moderately long calculation in which logarithms are used will give a result correct to *three* significant figures, or, in the case of an angle, to within 6′, if the tables are used in the way indicated in the preceding sections.

Further, if no use is made of the mean difference columns where "corrections" are given (see below), the result will in practically all cases be correct to *four* significant figures, or in the case of an angle to the nearest minute.

It will, however, be noted that at certain places in Clive's Mathematical Tables positive or negative decimal quantities are placed just above or below the decimals in the first ten

columns. For example, in Table I. such quantities are placed in the rows opposite the numbers

$$10, 11, 12, \ldots, 23.$$

These positive and negative decimal quantities are "correction quantities," and are used to give a more correct value to the mean difference.

If the "correction" be applied, as explained below, wherever in any calculation corrections are given, the final result will in general be correct to four (instead of to three) significant figures, and in the case of an angle to $1'$ instead of to $6'$.

In the case of Table I. the correction of the mean difference should be made as follows:—

Take the correction quantity given with the mantissa corresponding to the first three significant figures of the given quantity. Multiply it by the fourth significant figure. Take the result to the nearest integer and add it to the mean difference as found in § 180. This gives the *corrected mean difference.*

The method is shown in the following example:—

Ex. 1. To find log 1·759.

From the table $\log 1{\cdot}75 = {\cdot}24304$.

The correction quantity given, in this case, *above* ·24304 is $+{\cdot}2$.

Multiplying this by the fourth significant figure of 1·759, we get $+{\cdot}2 \times 9$ or 1·8, *i.e.* $+2$ to the nearest integer. The requisite mean difference is therefore

$$220+2 \text{ or } 222.$$

Thus	log 1·75 =	·24304
Corrected M.D. for	9 =	222
	log 1·759 =	·24526.

Ex. 2. To find antilog ·9748.

From the tables antilog $\cdot974 = 9{\cdot}4189$.

The correction quantity next to 9·4189 is given as $+{\cdot}1$.

The required correction is $+{\cdot}1 \times 8 = +{\cdot}8 = +1$ to nearest integer.

Thus	antilog ·974		= 9·4189
Corrected M.D. for	8 =	173+1 =	174
	∴ antilog ·9748		= 9·4363.

Ex. 3. To find antilog ·9907.

$$\text{Antilog } \cdot 990 = 9{\cdot}7724.$$

The correction quantity next to 9·7724 is given as $-\cdot 1$. Thus correction for mean difference

$$= -\cdot 1 \times 7 = -\cdot 7 = -1 \text{ to the nearest integer.}$$

$$\begin{array}{llr} \text{Thus} & \text{antilog } \cdot 990 & = 9{\cdot}7724 \\ \text{Corrected M.D. for} & 7 = 158 - 1 = & 157 \\ \hline & \therefore\ \text{antilog } \cdot 990\,7 & = 9{\cdot}7881. \end{array}$$

Ex. 4. To find $L \sin 7° 52'$.

$$L \sin 7° 48' = 9{\cdot}13263.$$

The correction for the mean difference for 4′ is

$$-1{\cdot}2 \times 4 = -4{\cdot}8 = -5 \text{ to nearest integer.}$$

$$\begin{array}{llr} \text{Thus} & L \sin 7° 48' & = 9{\cdot}13263 \\ \text{Corrected M.D. for} & 4' = 372 - 5 = & 367 \\ \hline & \therefore\ L \sin 7° 52' & = 9{\cdot}13630. \end{array}$$

The correction method can also be applied to L cos, L cot, and L cosec, which are worked from the *bottom* of the table. The method has some difficulties, so that in such cases when a correction is to be applied it is best to work by means of the function at the head of the table, as shown in the following example:—

Ex. 5. To find $L \cot 79° 7'$.

$$L \cot 79° 7' = L \tan (90° - 79° 7') = L \tan 10° 53'.$$

From the tables $L \tan 10° 48' = 9{\cdot}28049.$

The correction for the mean difference for 5′ is

$$-\cdot 6 \times 5 = -3.$$

$$\begin{array}{llr} \text{Thus} & L \tan 10° 48' & = 9{\cdot}28049 \\ \text{Corrected M.D. for} & 5' = 345 - 3 = & 342 \\ \hline & \therefore\ L \tan 10° 53' & = 9{\cdot}28391 \\ & \therefore\ L \cot 79° 7' & = 9{\cdot}28391. \end{array}$$

Ex. 6. To find cosec 5° 9′

$$L \operatorname{cosec} 5° 9' = L \sec (90° - 5° 9') = L \sec 84° 51'.$$

From the tables $L \sec 84° 48' = 11{\cdot}04272.$

The correction for the mean difference for 3′ is

$$+2{\cdot}6 \times 3 = +7{\cdot}8 = +8 \text{ to nearest integer.}$$

$$\begin{array}{llll} \text{Thus} & L \sec 84^\circ\, 48' & & = 11{\cdot}04272 \\ \text{Corrected M.D. for} & & 3' = 413+8 = & \phantom{11{\cdot}04}421 \\ \hline & \therefore\ L \sec 84^\circ\, 51' & & = 11{\cdot}04693 \\ & \therefore\ L \operatorname{cosec} 5^\circ\, 9' & & = 11{\cdot}04693. \end{array}$$

192. We will now illustrate the foregoing sections with some miscellaneous examples:—

Ex. 1. Find log versin $73^\circ\, 28'$.

Calling the angle $73^\circ\, 28' = A$, we have

$$\text{vers}\, A = 1 - \cos A = 2 \sin^2 \tfrac{1}{2}A,$$

$$\therefore\ \log \text{vers}\, A = \log 2 + 2 \log \sin \tfrac{1}{2}A.$$

Now $\qquad L \sin \frac{1}{2}(73^\circ\, 28') = L \sin 36^\circ\, 44'$.

$$\begin{array}{lrl} \text{From the tables} & L \sin 36^\circ\, 42' = & 9{\cdot}77643 \\ \text{diff. for} & 2' = & \phantom{9{\cdot}776}34 \\ \hline & \therefore\ L \sin 36^\circ\, 44' = & 9{\cdot}77677. \end{array}$$

$$\therefore\ \log \sin 36^\circ\, 44' = 9{\cdot}77677 - 10,$$

$$\therefore\ \log \text{vers}\, A = \log 2 + 2\,(9{\cdot}77677 - 10) = {\cdot}30103 + 19{\cdot}55354 - 20$$
$$= \bar{1}{\cdot}85457.$$

Ex. 2. To find to the nearest minute the angle between a diagonal and an edge of a cube.

Let a be the length of a side of the cube. Complete the right-angled triangle having the diagonal and an edge as hypotenuse and base. Then the length of the third side will be found to be $a\sqrt{2}$. Hence, if A be the required angle,

$$\tan A = \sqrt{2}.$$

$$\therefore\quad L \tan A = 10 + \log \sqrt{2} = 10 + \tfrac{1}{2} \log 2 = 10 + \tfrac{1}{2}\,({\cdot}30103)$$
$$= 10{\cdot}15051.$$

From the tables $\qquad L \tan 54^\circ\, 42' = 10{\cdot}14994$.

The difference between $10{\cdot}15051$ and $10{\cdot}14994 = {\cdot}00057$, and the nearest mean difference is ${\cdot}00053$, giving $2'$ as the difference of angle.

$$\therefore\ A = 54^\circ\, 42' + 2' = 54^\circ\, 44' \text{ to the nearest minute.}$$

Ex. 3. Find, to the nearest minute, the angle between the planes forming two adjacent faces of a regular tetrahedron.

It may be proved geometrically (and we shall assume) that the angle required is $\cos^{-1}(\frac{1}{3})$. Let θ be the angle.

Then
$$\begin{aligned} \cos\theta &= \tfrac{1}{3}, \\ L \cos\theta &= \log \tfrac{1}{3} + 10 \\ &= 10 - \log 3 \\ &= 10 - {\cdot}47712 \\ &= 9{\cdot}52288. \end{aligned}$$

From the tables $L \cos 70° 36' = 9{\cdot}52135$.

The difference between $9{\cdot}52135$ and $9{\cdot}52288 = {\cdot}00153$. The nearest mean difference in the same row is 145, giving a difference of angle equal to $4'$.

$$\therefore\ \theta = 70° 36' - 4'$$
$$= 70° 32' \text{ to the nearest minute.}$$

Ex. 4. Calculate in a decimal form the value of $\left(\frac{\sqrt{2}}{3}\right)^{\sqrt{3}}$

Let $$u = \left(\frac{\sqrt{2}}{3}\right)^{\sqrt{3}}$$

Then
$$\begin{aligned}
\log u &= \sqrt{3} \log \frac{\sqrt{2}}{3} \\
&= \sqrt{3}\,(\tfrac{1}{2}\log 2 - \log 3) \\
&= \sqrt{3}\,(\tfrac{1}{2} \times {\cdot}30103 - {\cdot}47712) \\
&= \sqrt{3}\,({\cdot}15051 - {\cdot}47712) \\
&= -\sqrt{3} \times {\cdot}32661.
\end{aligned}$$

Put this equal to $-p$.

Then
$$\begin{aligned}
p &= \sqrt{3} \times {\cdot}32661. \\
\therefore\ \log p &= \tfrac{1}{2}\log 3 + \log {\cdot}32661 \\
&= \tfrac{1}{2} \times {\cdot}47712 + \bar{1}{\cdot}51403 \\
&= \bar{1}{\cdot}75259. \\
\therefore\ p &= \text{antilog } \bar{1}{\cdot}75259 \\
&= {\cdot}56572. \\
\therefore\ \log u &= -{\cdot}56572 \\
&= \bar{1}{\cdot}43428. \\
\therefore\ u &= \text{antilog } \bar{1}{\cdot}43428 \\
&= {\cdot}27182.
\end{aligned}$$

$$\therefore\ \left(\frac{\sqrt{2}}{3}\right)^{\sqrt{3}} = {\cdot}27182.$$

EXAMPLES XVI.

1. Given $\log 2 = {\cdot}30103$, find the logarithm of 2000, $\frac{2}{100000}$, $\frac{1}{256}$; find also the logarithm of 10 to base 2.

2. Given $\log 2 = {\cdot}30103$, $\log 3 = {\cdot}47712$, $\log 7 = {\cdot}84510$, calculate $\log_{10}\left(\frac{75}{14}\right)$.

3. Find, from the tables, the logarithms of 316130, 316·25, 3168·6, ·031696.

4. Find the antilogarithms of $3{\cdot}50028$, $\bar{1}{\cdot}50114$.

5. Find, from 5-figure tables, the values of

(i) $L \sin 24° 58'$, (ii) $L \cos 24° 7'$,
(iii) $L \tan 65° 3'$, (iv) $L \sec 65° 58'$,
(v) $L \operatorname{cosec} 24° 5'$, (vi) $L \cot 65° 3'$.

6. Find, to the nearest minute, the angle whose

(i) $L \sin$ is 9·61046, (ii) $L \tan$ is 10·15028,
(iii) $L \cot$ is 9·86748, (iv) $L \operatorname{cosec}$ is 10·68956.

7. Find, by interpolation, the values of

$L \sin 3° 14'$,
$L \sin 2° 10'$,
$L \cos 86° 32'$.

8. Find, by interpolation, the values of

$L \tan 2° 28'$,
$L \tan 86° 51'$,
$L \cot 3° 22'$,
$L \cot 87° 50'$.

9. Find, by interpolation, the values of

$L \sec 86° 38'$,
$L \sec 87° 4'$,
$L \operatorname{cosec} 2° 35'$.

10. Given $\log 64{\cdot}14 = 1{\cdot}80713$, $\log 64{\cdot}15 = 1{\cdot}80720$, find by the principle of proportional parts the logarithm of log 64·147.

11. Given $\log 4{\cdot}52 = {\cdot}65514$, $\log 4{\cdot}53 = {\cdot}65610$, find by the principle of proportional parts the log of 4·522.

12. Find, by interpolation, to the nearest minute the angle whose

$L \sin$ is 8·75701,
$L \tan$ is 8·70135,
$L \sec$ is 11·15873.

13. Find, by interpolation, to the nearest minute the angle whose

$L \cos$ is 8·679,
$L \cot$ is 11·69,
$L \operatorname{cosec}$ is 11·33417.

14. Find the value of $\log \sin 30°$.

15. Calculate the values of $\sec 30°$, $\log_{10} \sec 30°$, and $L \sec 30°$.

16. Find the value of the seventh root of 100 to 4 decimal places.

17. Given $\log 7 = {\cdot}84510$, find the logarithm of

$$({\cdot}007)^{\frac{1}{4}} \div ({\cdot}07)^{3}.$$

18. Find the greatest and least values of $L \cot A$ as A changes from $0°$ to $90°$.

19. A number lies between two others whose difference is a small fraction of either. What assumption is made in obtaining an approximate expression for the logarithm of the first number in terms of the logarithms of the two others?

20. Given $L \cot 26° 10' = 10{\cdot}30862$, $L \cot 26° 20' = 10{\cdot}30543$, find by the method of proportional parts the value of A if $L \cot A = 10{\cdot}30734$.

21. Find x and y from the equations

$$\log x^3 + \log y^2 = 1{\cdot}85733,$$
$$\log x - \log y = \bar{1}{\cdot}82391.$$

22. Given $L \cos 24° 12' = 9{\cdot}96005$, $L \cos 24° 18' = 9{\cdot}95971$, calculate A approximately if $L \cos A = 9{\cdot}95992$.

23. Given $\log 2 = {\cdot}30103$, find the logarithms of $1000 \div 256$ and $1 \div 256$.

24. Write down the logarithms of 5374·5, 5374500, and ·0053745.

25. Find the fifth root of $\cdot 0002 \div 23087$.

26. Find the logarithm of $97{\cdot}942 \times {\cdot}0063864$ and the seventh root of $\frac{13}{300}$.

27. Calculate $\sqrt[5]{\cdot 02}$, and $\dfrac{1}{(1{\cdot}5866)^3}$.

28. Find the logarithm of the tangent of $81° 11'$, and, to the nearest minute, the angle whose L cotangent is 9·61705.

29. Find the numerical value of $\sqrt[3]{(\tan 50° \tan 22° 30')}$.

30. Find $\log \tan 35° 16'$ and the numerical value of $\sqrt[3]{(\frac{1}{3} \sin 44°)}$.

31. Find the value of $\sqrt[5]{(\tan 40° \div 65)}$.

32. Find the numerical value of the seventh root of $[(\tan 53° 30') \div (32)]$.

33. Calculate the numerical value of the following expressions:—

(*a*) $\cos 27° 29' \cos 172° 9'$; (*b*) $\cos 70° 22' - \cos 54° 40'$.

34. Calculate in a decimal form $(\sin 60°)^{\sin 60°}$.

35. Find by interpolation the value of $L \tan 3° 15'$, and calculate the cube root of the tangent.

36. Calculate to the nearest minute the angle whose sine is $\frac{1}{3}$.

37. Take out from the tables the L tangent of $16° 7'$, and calculate the value of the square root of the tangent.

38. Solve the equation $13 \sin \theta = 3$.

39. Solve the equation $\cos\theta = \sin 49^\circ 26' - \sin 75^\circ 58'$.

40. Find all the values of θ less than 180° which satisfy the equations:
(*a*) $17\sin\theta = 15\sin 63^\circ 18'$; (*b*) $\cos\theta = \cos 37^\circ 59' \cos 153^\circ 18'$;
(*c*) $\tan 2\theta = -\sin 52^\circ 2'$; (*d*) $\cos^3\theta = \cos 73^\circ 6' - \cos 17^\circ 42'$.

41. If $a = 5$ inches and $c = a\tan 32^\circ 15'$, calculate the value of

$$2\sqrt{\frac{a^2+ac+c^2}{2}}.$$

42. Calculate $(\frac{1}{3}\pi)^{\frac{1}{10}}$, π being $3{\cdot}1416$.

CHAPTER XVII.

LOGARITHMIC SOLUTION OF RIGHT-ANGLED TRIANGLES.

193. In this chapter we shall exemplify the use of logarithmic methods by applying them to the solution of right-angled triangles.

In Chapter III. we considered the solution of right-angled triangles without the aid of logarithms, and employing no trigonometric functions but the sine, cosine, and tangent. In addition to the formulae written down in § 27, six others practically equivalent to them might be obtained by writing down the secants, cosecants, and cotangents of the two acute angles in terms of the sides. It is, however, undesirable to remember special formulae for right-angled triangles, as they may be at once written down from a figure.

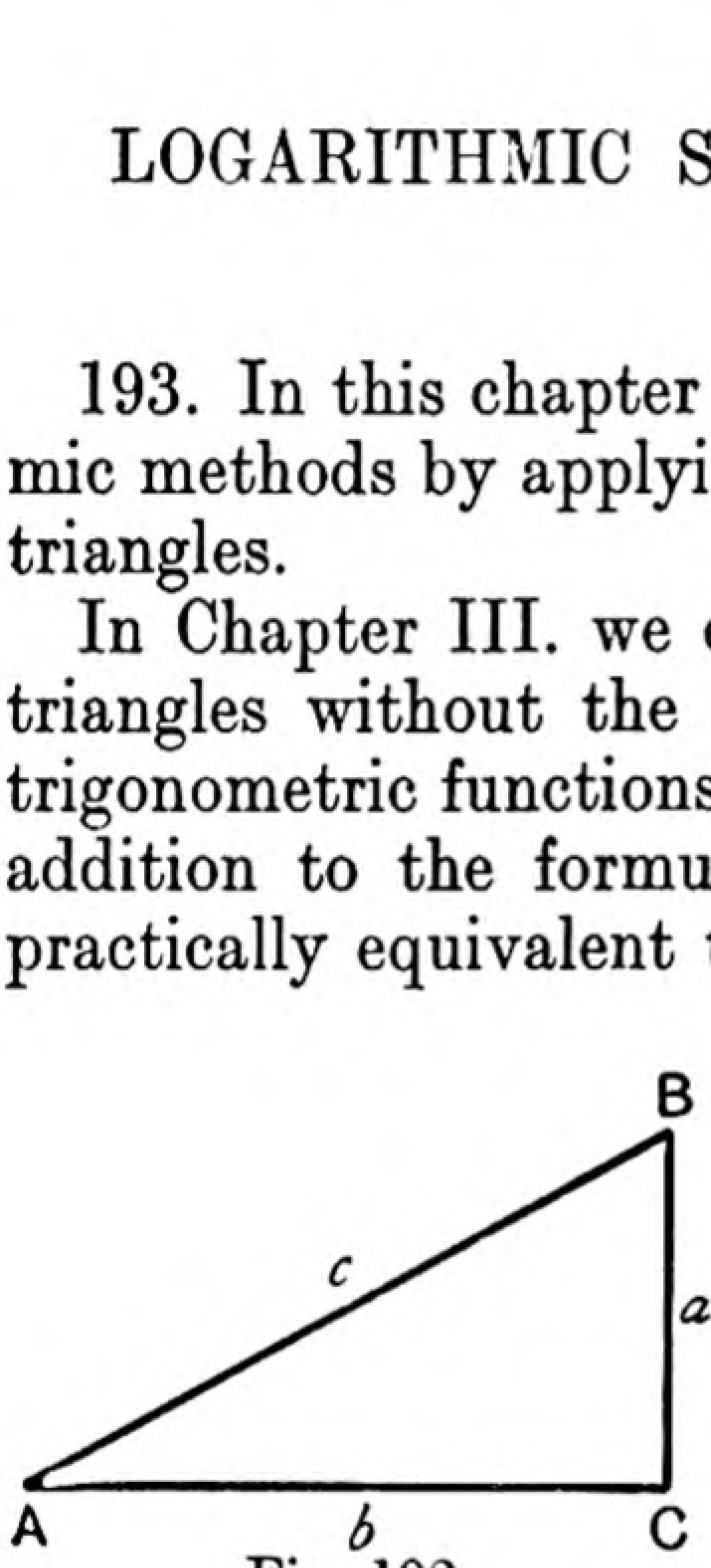


Fig. 102.

The notation in this and succeeding chapters will be the same as explained in § 26, viz. A, B, C will denote the angles and a, b, c the opposite sides of the triangle. In the present chapter C will denote the right angle except where otherwise specified, but in the solution of examples the student will be required to use any notation that may be proposed.

A few general hints may prove useful to the reader.

194. **The relation connecting any two sides and an angle** may be at once written down by expressing the ratio of the sides as a trigonometric function of the given angle. In

putting this relation into logarithmic form, attention must be paid to the rules for the logarithm of a product or a quotient, and the difference of 10 between the actual and tabular logarithms must be allowed for.

Ex. Thus, to find the logarithmic relations between a, c, and B, we at once write down from the figure

$$\cos B = a/c, \qquad \sec B = c/a;$$

$$\therefore \quad \log\cos B = \log a - \log c \text{ and } \log\sec B = \log c - \log a;$$

$$\therefore \quad L\cos B - 10 = \log a - \log c \text{ and } L\sec B - 10 = \log c - \log a;$$

either of which may be used to find the third of the quantities b, c, A, when two of them are known.

When one of the angles is known, the other should be at once found from the relation

$$A + B = 90°.$$

When the two sides containing the right angle are given, the hypotenuse cannot be calculated directly by logarithms, for the formula $c^2 = a^2 + b^2$ is not adapted to logarithmic calculation when c is required. There is no formula for the logarithm of a *sum*, and hence the logarithm of the sum of the squares of a and b cannot be expressed in terms of $\log a$ and $\log b$.

We therefore find one of the angles first and then find the hypotenuse, using, *e.g.* the equations—

$$\tan A = a/b,$$

and then $$c = b\sec A \quad \text{or} \quad c = a\operatorname{cosec} A,$$

which are adapted to logarithmic computation.

When one side and the hypotenuse are given, it is still best to determine the angles first if their values are required. If not, the remaining side b may be found from the formula

$$b^2 = c^2 - a^2 = (c+a)(c-a),$$

which may be put into the logarithmic form

$$2\log b = \log(c+a) + \log(c-a),$$

so that we must take from the tables the logarithms of $c+a$ and $c-a$, not those of c and a.

Ex. 1. The sides of a right-angled triangle ($C = 90°$) are

$$a = 5,\ b = 12;\ \text{find } A.$$

The relation connecting A with a, b is

$$a/b = \tan A;$$

$$\therefore\ \log a - \log b = L \tan A - 10,$$

$$\therefore\ \log 5 - \log 12 = L \tan A - 10,$$

$$\therefore\ L \tan A = 1 - \log 2 - \log 12 + 10 = 11 - 3 \log 2 - \log 3$$

$$= 11 - {\cdot}90309 - {\cdot}47712 = 9{\cdot}61979.$$

Now $$L \tan 22° \, 36' = 9{\cdot}61936.$$

The difference between $L \tan 22° \, 36'$ and $L \tan A$

$$= 9{\cdot}61979 - 9{\cdot}61936 = {\cdot}00043.$$

The nearest mean difference is ·00036, giving a difference of angle equal to $1'$,

$$\therefore\ A = 22° \, 37' \text{ to nearest minute.}$$

Ex. 2. Find B in a right-angled triangle, having given

$$c = 16,\ b = 4,\ C = 90°.$$

The relation connecting B with b and c is

$$b/c = \sin B,$$

$$\therefore\ \log b - \log c = L \sin B - 10,$$

$$\therefore\ L \sin B = \log 4 - \log 16 + 10 = 10 - 2 \log 2$$

$$= 10 - {\cdot}60206 = 9{\cdot}39794.$$

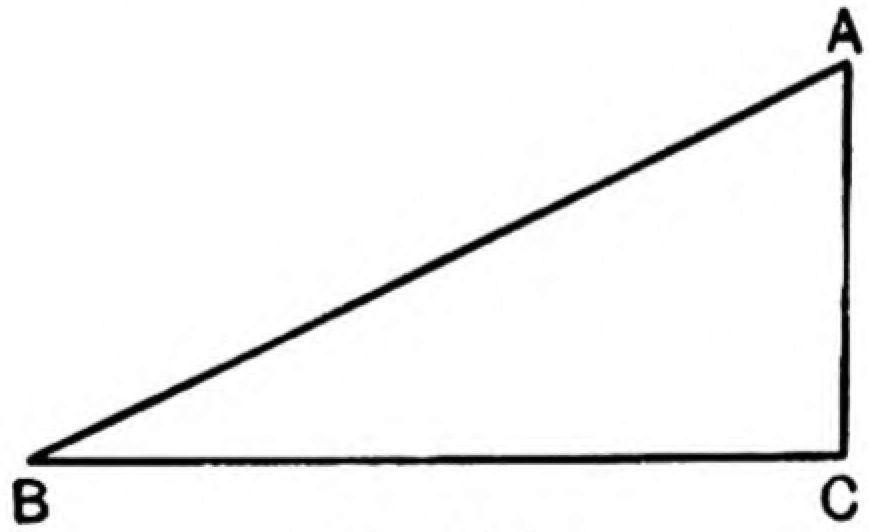


Fig. 103.

Now $$L \sin 14° \, 24' = 9{\cdot}39566.$$

The difference between $L \sin 14° \, 24'$ and $L \sin B$

$$= 9{\cdot}39794 - 9{\cdot}39566 = {\cdot}00228.$$

The nearest mean difference is ·00249, giving a difference of angle equal to $5'$,

$$\therefore\ B = 14° \, 29' \text{ to nearest minute.}$$

Ex. 3. Find A in a right-angled triangle, having given

$$c = 93{\cdot}7,\ a = 1{\cdot}3,\ C = 90^\circ.$$

The relation connecting A with a and c is

$$\begin{aligned} a/c &= \sin A, \\ \therefore\quad L\sin A &= 10+\log 1{\cdot}3-\log 93{\cdot}7, \\ &= 10{\cdot}11394-1{\cdot}97174, \\ &= 8{\cdot}14220. \end{aligned}$$

On referring to the tables we find we must interpolate to determine A.

$$\begin{aligned} L\sin 42' &= 8{\cdot}08696, \\ L\sin 48' &= 8{\cdot}14495, \\ \hline \text{Diff. for } 6' &= \ {\cdot}05799. \end{aligned}$$

Let $\qquad A = 42'+x'.$

Then $\qquad L\sin(42'+x') = 8{\cdot}14220,$

and $\qquad L\sin 42' = 8{\cdot}08696,$

$$\therefore\quad \text{diff. for } x' = {\cdot}05524,$$

$\therefore$ by the theory of proportional parts

$$\frac{x'}{6'} = \frac{{\cdot}05524}{{\cdot}05799},$$

$$\therefore\quad x = \frac{5524}{5799}\times 6' = 5{\cdot}7',$$

$\therefore$ to the nearest minute $x' = 6'$,

$$\therefore\quad A = 48' \text{ to the nearest minute.}$$

Ex. 4. Given $C = 90^\circ$, $a = 117{\cdot}24$, and $b = 236{\cdot}28$ (feet in each case), find by logarithms the length c, also expressed in feet.

Since the relation $c^2 = a^2+b^2$ is not adapted to logarithms, we must find one of the angles first. We have

$$\tan A = a/b, \qquad \sec A = c/b;$$

$$\therefore\quad \log\tan A = L\tan A-10 = \log a-\log b$$

and $\qquad \log c = \log b+\log\sec A = \log b+L\sec A-10;$

$$\begin{aligned} \therefore\quad L\tan A &= 10+\log 117{\cdot}24-\log 236{\cdot}28 \\ &= 12{\cdot}06908 \qquad\qquad \left.\begin{array}{r}(10+\log a)\\ (\log b)\end{array}\right\}\begin{array}{l}\text{from}\\ \text{tables}\end{array} \\ &\quad\ -2{\cdot}37343 \\ &= \ \ 9{\cdot}69565 \end{aligned}$$

Now $$L \tan 26^\circ 18' = 9{\cdot}69393.$$

The difference between $L \tan 26^\circ 18'$ and $L \tan A$

$$= 9{\cdot}69565 - 9{\cdot}69393 = {\cdot}00172.$$

This is greater than the mean difference for $5'$, viz. $\cdot 00158$.

Thus A is greater than $26^\circ 23'$ and may be nearer to $26^\circ 24'$ than to $26^\circ 23'$.

On examination, $$L \tan 26^\circ 23' = 9{\cdot}69551$$

and $$L \tan 26^\circ 24' = 9{\cdot}69584.$$

Hence $L \tan A$ is nearer to $L \tan 26^\circ 23'$ than to $L \tan 26^\circ 24'$,

$$\therefore \quad A = 26^\circ 23' \text{ to the nearest minute.}$$

Now
$$\begin{aligned} \log c &= \log b + L \sec A - 10 \\ &= 2{\cdot}37345 + 10{\cdot}04778 - 10 \\ &= 2{\cdot}42123, \\ \therefore \ c &= \text{antilog } 2{\cdot}42123 \\ &= 263{\cdot}77 \text{ feet.} \end{aligned}$$

195. **Completion of the solution.**—Excluding the right angle, a right-angled triangle has five parts, and two of these parts are sufficient to determine the triangle provided that one at least of the given parts is a side. To solve the triangle *completely* the three remaining parts have all to be found. One of these three parts is an acute angle which is connected with the other acute angle by the relation $A + B = 90^\circ$. Hence, in every case, *two and only two parts have to be calculated with the aid of logarithms.*

In the second calculation it is convenient, as far as possible, to make use of logarithms that have already been used in the first calculation.

196. Problems are often proposed in which the given data require a method of solution to be adopted different from that which would naturally be employed in working with tables.

Thus, *e g.* if two sides of a right-angled triangle are given whose ratio can be resolved into products of powers of small numbers like 2, 3, or 5, it may be required to solve the triangle having given log 2 and log 3.

In such cases common sense will alone indicate the right method to pursue, as such problems are proposed expressly to test the ingenuity of the student.

197. *Ex.* Prove that $\tan \frac{1}{2}(A-B) = \dfrac{a-b}{a+b}$,

and, if $$a = 22,\ b = 103,\ C = 90°,$$
find A, B, having given
$$\log 2 = \cdot 30103, \quad \log 3 = \cdot 47712.$$

(i) $\tan \frac{1}{2}(A-B) = \tan \frac{1}{2}(90° - B - B) = \tan(45° - B)$
$$= \frac{1-\tan B}{1+\tan B} = \frac{1-b/a}{1+b/a} = \frac{a-b}{a+b} \text{ (since } \tan B = b/a).$$

(ii) Here $b>a$, and therefore we must avoid negative quantities by taking the corresponding form with A, B interchanged, thus
$$\tan \tfrac{1}{2}(B-A) = \frac{b-a}{b+a} = \frac{103-22}{103+22} = \frac{81}{125} = \frac{3^4}{5^3};$$
$$\therefore \log \tan \tfrac{1}{2}(B-A) = 4\log 3 - 3\log 5 = 4\log 3 - 3(\log 10 - \log 2)$$
$$= 4\log 3 + 3\log 2 - 3;$$
$$\therefore L \tan \tfrac{1}{2}(B-A) = 4 \times (\cdot 47712) + 3 \times (\cdot 30103) + 7$$
$$= 1\cdot 90848 + \cdot 90309 + 7 = 9\cdot 81157;$$
$\therefore$ from the tables
$$\tfrac{1}{2}(B-A) = 32°\ 57' \text{ to nearest minute.}$$
But $$\tfrac{1}{2}(B+A) = 45°.$$
Hence by adding and subtracting
$$B = 77°\ 57', \quad A = 12°\ 3'.$$

EXAMPLES XVII.

Solve, by means of 5-figure tables, the triangles (1–6), in which C is the right angle:—

1. Given $a = 837\cdot 21$, $b = 694\cdot 73$;
2. Given $A = 20°\ 14'$, $b = 4930$;
3. Given $c = 840$, $A = 38°\ 16'$;
4. Given $c = 726\cdot 9$, $b = 316\cdot 2$;
5. Given $a = 123\cdot 45$, $b = 234\cdot 56$;
6. Given $a = \cdot 04$, $A = 40°$;

Solve the following triangles (7–12):—

7. Given $A = 52°\ 38'$, $b = 45$, $B = 90°$;
8. Given $A = 49°\ 14'$, $c = 331$, $B = 90°$;
9. Given $A = 56°\ 29'$, $b = 4264\cdot 3$, $B = 90°$;
10. Given $A = 4°\ 44'$, $a = 694\cdot 73$, $B = 90°$;

11. Given $c = \cdot 2$, $A = 40°$, $B = 90°$;

12. Given $b = 1777 \cdot 5$, $c = 1177$, $A = 90°$;

13. Given $c = 6 \cdot 953$, $b = 3$, $C = 90°$; find B.

14. **ABC** is a triangle with a right angle at **C**, **CB** is 30 ft. long, and **BAC** is 20°. If **CB** is produced to a point **P** such that **PAC** = 55°, calculate the length of **CP**.

15. The elevation of a tower is observed from two points in the same horizontal line with its base, and the distance between the points of observation is known. Investigate a formula for the height of the tower. Calculate it from the following data: angles 20° and 55°; distance between the points of observation, 1,000 ft.

16. In a triangle **ABC**, the base **AB** is 1,000 ft. long, and the angles at **A** and at **B** are 31° 20′ and 125° 19′, respectively: find the length of the perpendicular let fall from **C** on **AB** produced and the distance from **A** to the foot of the perpendicular.

17. **AB** is a horizontal line 1,300 ft. long. A vertical line is drawn from **B** upwards, and in it two points **P** and **Q** are taken such that **BQ** is three times **BP**; **BAP** is 10° 30′. Calculate **BP** and **BAQ**.

18. Find the angles of a right-angled triangle **ABC**, having given that the base **AC** is 15,866 ft. and the height **BC** 13,000 ft. Find also the length of the line drawn from **B** to **AC** which bisects the angle **ABC**.

19. The base of an isosceles triangle **ABC** is 1,300 ft. long, and the altitude is double that of an equilateral triangle on an equal base; the angles A, B, C are bisected by lines which meet in **D**. Find to the nearest minute the angle **DAB**, and the number of square feet in the area of the triangle **DBC**.

20. The summit of a spire is vertically over the middle point of a square enclosure whose side is a ft. long; the height of the spire is h ft. above the level of the square. If the shadow of the spire just reaches a corner of the square when the sun has an altitude θ, show that

$$h\sqrt{2} = a \tan \theta.$$

Calculate h; given $a = 1{,}000$ ft., $\theta = 27° \, 30'$.

21. **AB** is a vertical pole 50 ft. high, **A** being above **B**; **BC** is inclined upwards at an angle of 20° to the horizontal line **BD**, so that **ABC** is an angle of 70°; the shadow of **AB** on **BC** is 50 ft. long. If the shadow fall on **BD**, what would be its length ?

22. The angular elevation of a tower at a certain station is A; at another station, in the same horizontal plane and a ft. nearer the tower, the angular elevation is $90° - A$. If h be the height of the tower, show that

$$h\,(1 - \tan^2 A) = a \tan A.$$

Calculate h, when $A = 10°$ and $a = 100$ ft.

23. Find the angle A of the triangle **ABC**, having given that **AC** $=$ 257 ft., **AB** $=$ 650 ft., and $C = 90°$. Find also the length of the line **AD** which meets **BC** in **D**, so that the angle **ADC** $= 40° 32'$.

Solve the following triangles:—

24.	$C = 90°$,	$A = 50° 11'$,	$a = 65{\cdot}89$.
25.	$A = 90°$,	$B = 62° 18'$,	$b = 130{\cdot}5$.
26.	$B = 90°$,	$C = 17° 36'$,	$c = 762{\cdot}3$.
27.	$C = 90°$,	$A = 32° 13'$,	$a = 16{\cdot}83$.
28.	$B = 90°$,	$A = 65° 18'$,	$a = 544{\cdot}7$.
29.	$A = 90$,	$C = 78° 10'$,	$c = 108{\cdot}5$.
30.	$A = 90°$,	$B = 54° 18'$,	$c = 96{\cdot}34$.
31.	$A = 90°$,	$B = 12° 16'$,	$c = 73{\cdot}69$.
32.	$B = 90°$,	$A = 70° 10'$,	$c = 11{\cdot}46$.
33.	$B = 90°$,	$C = 15° 15'$,	$a = 62$.
34.	$B = 90°$,	$C = 44° 10'$,	$a = 365$.
35.	$C = 90°$,	$A = 38° 16'$,	$b = 50$.
36.	$C = 90°$,	$A = 56° 10'$,	$c = 963$.
37.	$C = 90°$,	$B = 13° 13'$,	$c = 1000$.
38.	$A = 90°$,	$B = 10° 38'$,	$a = 27$.
39.	$A = 90°$,	$B = 45°$,	$a = 500$.
40.	$A = 90°$,	$C = 60°$,	$b = 360$.
41.	$B = 90°$,	$C = 30°$,	$a = 700$.
42.	$B = 90°$,	$C = 75°$,	$a = 515$.
43.	$B = 90°$,	$C = 65°$,	$a = 670$.

CHAPTER XVIII.

TRIGONOMETRIC PROPERTIES OF TRIANGLES IN GENERAL.

198. **Number of data required to fix a triangle.**—Before proceeding to apply logarithms to the solution of oblique-angled triangles, *i.e.* triangles other than right-angled, it will be necessary to establish certain relations connecting the sides of any triangle with the trigonometrical functions of its angles themselves or of angles related to them.

A triangle can, in general, be constructed geometrically, and is therefore theoretically completely determined if we are given any three of its six parts (*i.e.* sides and angles), provided that at least one of these parts is a *side*. The given parts may therefore be either—

(i) One side and two angles.
(ii) Two sides and the angle opposite one of them.
(iii) Two sides and the included angle.
(iv) All three sides.

199. *To prove* that these data are really sufficient to fix the triangle, it is only necessary to prove that any two triangles having the given data are equal in all respects. This is proved for Case (i) in Euclid I. 26, for Case (iii) in Euclid I. 4, and for Case (iv) in Euclid I. 8. In Case (ii) we shall see that *two* triangles may *sometimes* be constructed having the given data; but, by following the methods of proof of Euclid VI. 7, it may be proved that any *other* triangle having the given parts is equal in all respects with one of these.

If the three angles alone were given, an infinite number of triangles could be constructed, provided that the given angles satisfied the relation $A+B+C = 180°$. For, if one such triangle were constructed, any triangle similar to it would have the same three angles.

It will be noted that all the following proofs depend on the property that any triangle may be divided into two right-angled triangles by letting fall a perpendicular from one vertex on the opposite side:—

200. The sides of any triangle are proportional to the sines of the opposite angles; or, in other words—

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{.................}(103)$$

One of the angles of the triangle, say B, will be acute. C may then be acute, obtuse, or right.

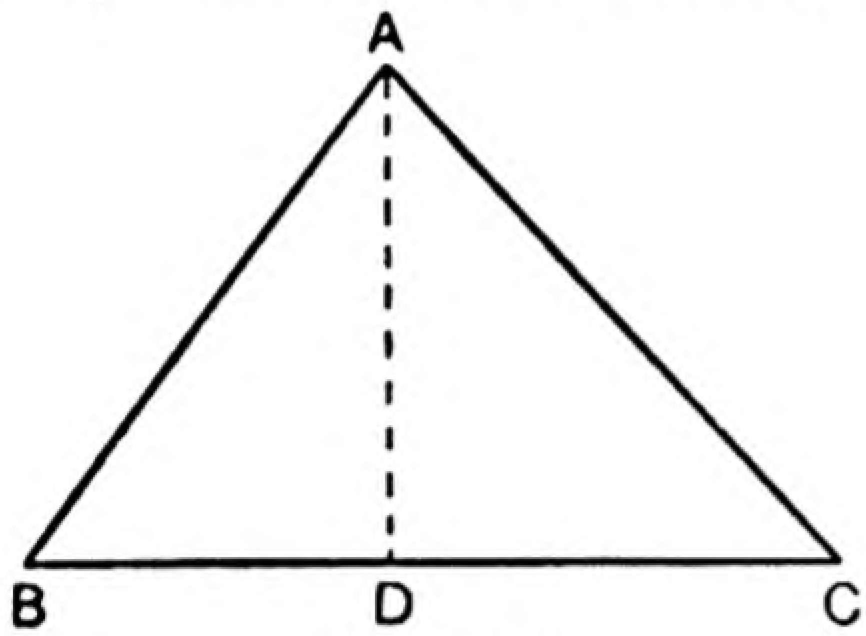


Fig. 104.

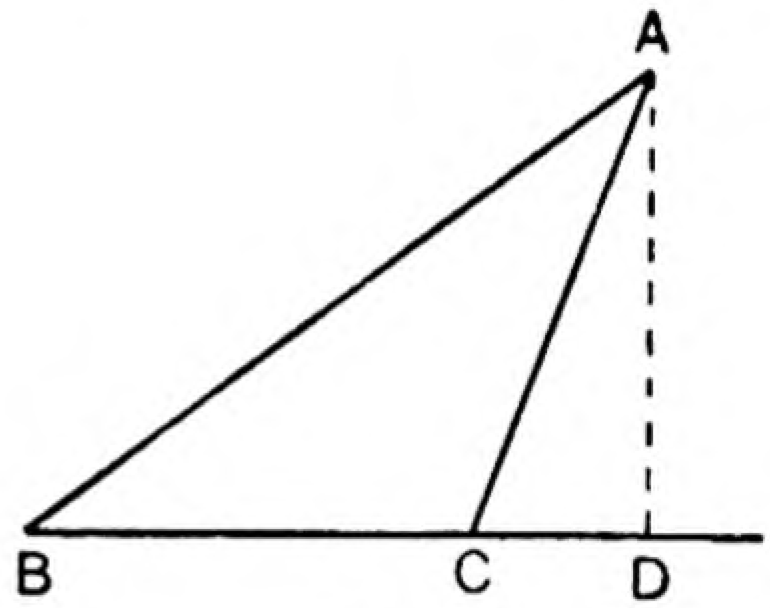


Fig. 105.

Draw **AD** perpendicular to the base, or base produced. Then, since B is acute,

$$\mathbf{DA} = \mathbf{BA} \sin B = c \sin B.$$

If C be acute (Fig. 104),

$$\mathbf{DA} = \mathbf{CA} \sin C = b \sin C.$$

If C be obtuse (Fig. 105),

$$\mathbf{DA} = \mathbf{CA} \sin \mathbf{ACD} = b \sin (180° - C) = b \sin C.$$

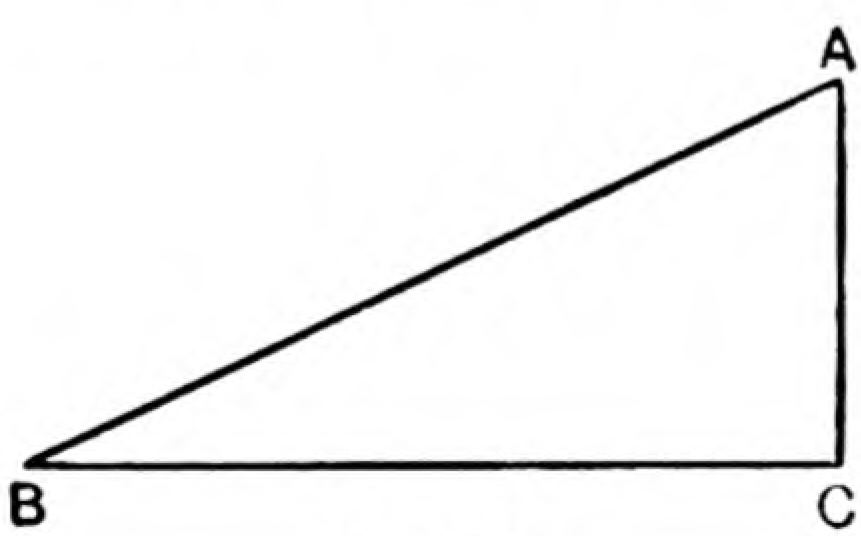


Fig. 106.

If C be right (Fig. 106), D will coincide with C, and hence

$$\mathbf{DA} = \mathbf{CA} = \mathbf{CA} \sin C = b \sin C$$

(since $\sin C = \sin 90° = 1$).

Hence, in each case,

$$c \sin B = b \sin C;$$

or

$$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

It may be similarly proved or deduced from the Principle of Symmetry (§ 146) that

$$\frac{\sin A}{a} = \frac{\sin B}{b};$$

$$\therefore \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{...............(103)}$$

These relations are sometimes known as the **sine rule.** They enable us to replace any relation connecting the sides of a triangle by a corresponding relation involving the sines of its angles.

201. *Ex.* 1. If $\sin^2 A + \sin^2 B = \sin^2 C$, find C.

Since the sines are proportional to the opposite sides,

$$\therefore \quad a^2 + b^2 = c^2.$$

Hence, by Euclid I. 48, $\qquad C = 90°.$

Ex. 2. The sides of a triangle a, b, c being in Arithmetical Progression, to find a limit to the angle opposite the mean side b.

Here $\qquad 2b = a + c;$

also $\sin A$, $\sin B$, $\sin C$ are proportional, respectively, to a, b, c.

$$\therefore \quad 2 \sin B = \sin A + \sin C;$$

$$\therefore \quad 4 \sin \frac{B}{2} \cos \frac{B}{2} = 2 \sin \frac{A+C}{2} \cos \frac{A-C}{2}$$

$$= 2 \cos \frac{B}{2} \cos \frac{A-C}{2};$$

$$\therefore \quad \sin \frac{B}{2} = \frac{1}{2} \cos \frac{A-C}{2}.$$

But $\qquad \cos \frac{1}{2}(A-C) < 1;$

$$\therefore \quad \sin \tfrac{1}{2}B < \tfrac{1}{2}; \quad \therefore \quad \tfrac{1}{2}B < 30°;$$

$$\therefore \quad B < 60°;$$

except when $A - C = 0$. In this case, $B = 60°$, and the triangle is equilateral. Therefore the limit is $60°$.

202. To express the cosines of the angles of a triangle in terms of the sides.—We shall first establish the formulae

$$\left.\begin{aligned} a^2 &= b^2+c^2-2bc\cos A \\ b^2 &= c^2+a^2-2ca\cos B \\ c^2 &= a^2+b^2-2ab\cos C \end{aligned}\right\} \quad \text{..................(104)}$$

Suppose it is required to prove the formula involving $\cos C$, *i.e.* the third of the above formulae.

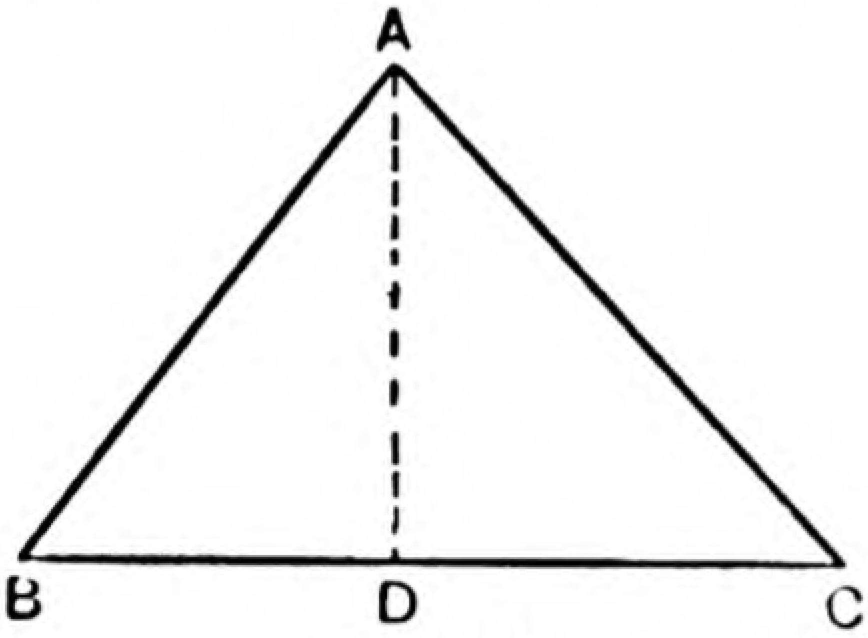


Fig. 107.

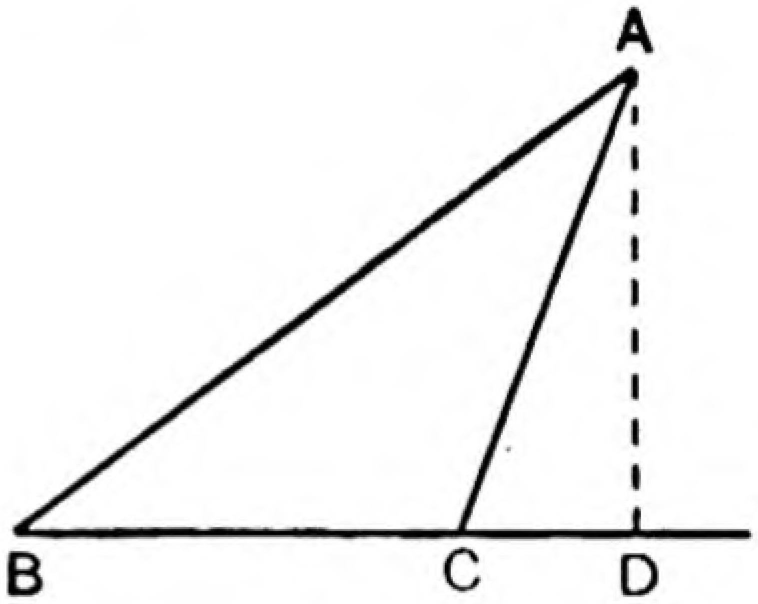


Fig. 108.

Whether C is acute or obtuse, one, if not both, of the other angles A, B will be acute. Let B be acute, and from the third vertex **A** draw **AD** perpendicular on **BC** or **BC** produced. Then, as in § 201, we have in all cases,

$$\mathbf{DA} = b\sin C.$$

If C be acute (Fig. 107),

$$\mathbf{CD} = \mathbf{CA}\cos C = b\cos C;$$

$$\therefore \quad \mathbf{BD} = \mathbf{CB}-\mathbf{CD} = a-b\cos C.$$

If C be obtuse (Fig. 108),

$$\mathbf{CD} = \mathbf{CA}\cos \mathbf{DCA} = b\cos(180^\circ-C);$$

$$\therefore \quad \mathbf{BD} = \mathbf{BC}+\mathbf{CD} = a+b\cos(180^\circ-C)$$

$$i.e.\ \mathbf{BD} = a-b\cos C.$$

If C be right (Fig. 106, § 201),

$$\mathbf{CD} = 0 = b\cos C \text{ (since } \cos C = \cos 90^\circ = 0);$$

$$\therefore \quad \mathbf{BD} = \mathbf{BC}-0 = a-b\cos C.$$

In all cases, therefore,

$$\mathsf{AB}^2 = \mathsf{BD}^2 + \mathsf{AD}^2 = (a-b\cos C)^2 + b^2\sin^2 C$$
$$= a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C;$$
$$\therefore \quad c^2 = a^2 - 2ab\cos C + b^2.$$

The other formulae may be proved similarly, or their truth inferred from the Principle of Symmetry.

The three formulae may next be written in the forms

$$\left.\begin{array}{l} \cos A = \dfrac{b^2+c^2-a^2}{2bc}, \quad \cos B = \dfrac{c^2+a^2-b^2}{2ca} \\ \qquad\qquad \cos C = \dfrac{a^2+b^2-c^2}{2ab} \end{array}\right\} \ldots(104\text{ A})$$

and from these, if we know the three sides of a triangle, we can calculate the cosines of its angles.

These three formulae constitute what is sometimes known as the **cosine rule.**

Ex. The sides **BC, CA, AB** of a triangle **ABC** are 2, 3, 4 in. long, respectively. Find the cosines of its angles.

Here $a = 2$, $b = 3$, $c = 4$; hence, by the formulae,

$$\cos A = \frac{3^2+4^2-2^2}{2.3.4} = \frac{9+16-4}{24} = \tfrac{21}{24} = \tfrac{7}{8}.$$

$$\cos B = \frac{4^2+2^2-3^2}{2.4.2} = \frac{16+4-9}{16} = \tfrac{11}{16}.$$

$$\cos C = \frac{2^2+3^2-4^2}{2.2.3} = \frac{4+9-16}{12} = -\tfrac{3}{12} = -\tfrac{1}{4}.$$

Illustrative Exercise.

If two adjacent sides of a parallelogram are a and b, and the included angle is C, prove that the length of the diagonal (d) through that angle is given by $$d^2 = a^2 + b^2 + 2ab\cos C.$$

203. Alternative proof of the formula

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The formulae of the preceding article can be obtained more briefly by making use of Euclid II. 12 and 13.

Make the same construction as before. If C be obtuse, then

$$\mathsf{BA}^2 = \mathsf{BC}^2 + \mathsf{CA}^2 + 2\mathsf{BC}.\mathsf{CD}. \qquad \text{(Euc. II. 12)}$$

But $$\mathbf{CD} = \mathbf{CA}\cos(180^\circ - C);$$

$$\therefore \quad c^2 = a^2+b^2+2ab\cos(180^\circ - C),$$

$$c^2 = a^2+b^2-2ab\cos C.$$

If C be acute, then

$$\mathbf{BA}^2 = \mathbf{BC}^2+\mathbf{CA}^2-2\mathbf{BC}.\mathbf{DC}, \qquad \text{(Euc. II. 13)}$$

and $$\mathbf{DC} = \mathbf{AC}\cos C;$$

$$\therefore \quad c^2 = a^2+b^2-2ab\cos C.$$

If C be a right angle, then

$$\mathbf{BA}^2 = \mathbf{BC}^2+\mathbf{CA}^2; \qquad \text{(Euc. I. 47)}$$

$$\therefore \quad c^2 = a^2+b^2 = a^2+b^2-2ab\cos C, \text{ since } \cos C = 0.$$

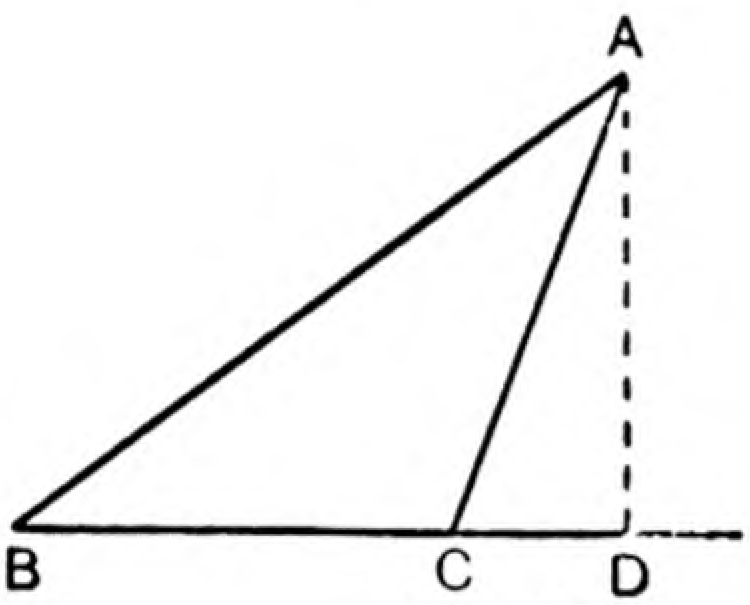


Fig. 109.

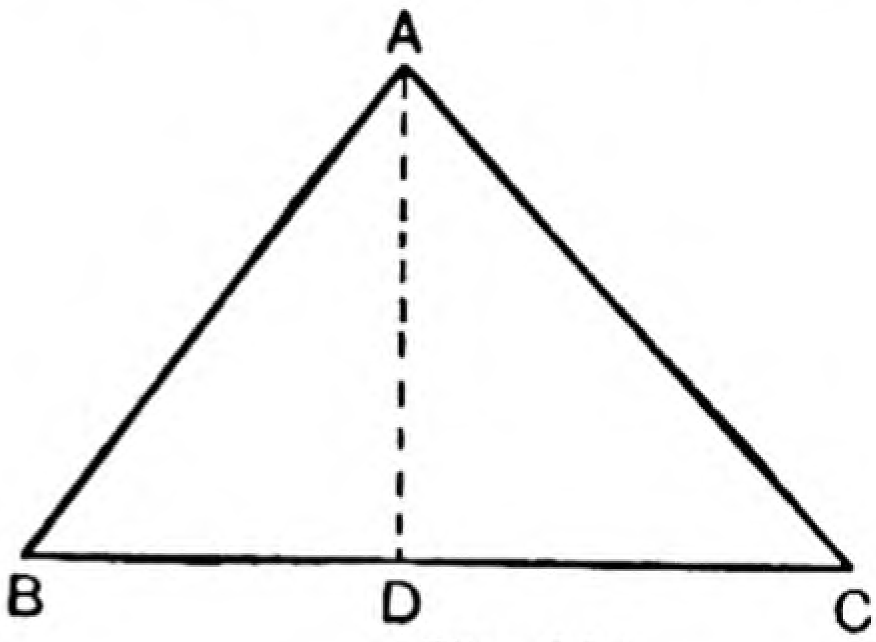


Fig. 110.

204. To prove the formula

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} \quad \text{.................(105)}$$

By the sine rule, $$\frac{a}{b} = \frac{\sin A}{\sin B};$$

hence, by a well-known theorem in proportion,*

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}$$

$$= \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)};$$

* If $\frac{a}{b} = \frac{c}{d}$, then, evidently, $\frac{a/b-1}{a/b+1} = \frac{c/d-1}{c/d+1}$, and therefore $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

$$\therefore \quad \tan\frac{A-B}{2} = \frac{a-b}{a+b}\tan\frac{A+B}{2} = \frac{a-b}{a+b}\tan\left(90^\circ - \frac{C}{2}\right)$$

$$= \frac{a-b}{a+b}\cot\frac{C}{2} \quad \text{.............................(105)}$$

This result is known as **Napier's Analogy**; it may also be called the **tangent rule.** The form

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}$$

is a convenient form for remembering.*

Taking logarithms, we have

$$\log\tan\tfrac{1}{2}(A-B) = \log(a-b) - \log(a+b) + \log\cot\tfrac{1}{2}C;$$

or, adding 10 to both sides to reduce to *tabular* logarithms,

$$L\tan\tfrac{1}{2}(A-B) = \log(a-b) - \log(a+b) + L\cot\tfrac{1}{2}C,$$

which is a form adapted for logarithmic computation.

205. To prove that

$$\boldsymbol{a = c\cos B + b\cos C} \quad \text{...................(106)}$$

With the same constructions, we have, if both the angles B, C are acute,

$$\mathbf{BC} = \mathbf{BD} + \mathbf{DC} = \mathbf{BA}\cos\mathbf{DBA} + \mathbf{AC}\cos\mathbf{DCA};$$

$$\therefore \quad a = c\cos B + b\cos C.$$

If C be obtuse,

$$\mathbf{BC} = \mathbf{BD} - \mathbf{CD} = \mathbf{BA}\cos\mathbf{CBA} - \mathbf{CA}\cos\mathbf{DCA}$$

$$\therefore \quad a = c\cos B - b\cos(180^\circ - C) = c\cos B + b\cos C.$$

In like manner, or from considerations of symmetry,

$$b = a\cos C + c\cos A,$$

$$c = b\cos A + a\cos B.$$

* "Analogy" is an old-fashioned name for a proportion. The result was stated by Napier in the form of the proportion

$$\tan\tfrac{1}{2}(A-B) : \tan\tfrac{1}{2}(A+B) = a-b : a+b.$$

Ex. To deduce the relations

$$a = c \cos B + b \cos C, \quad c^2 = a^2 + b^2 - 2ab \cos C,$$

from the sine rule and the relation $A + B + C = 180°$.

Since $A = 180° - (B + C)$;

$$\therefore \quad \sin A = \sin B \cos C + \cos B \sin C \quad \text{.....................(i)}$$

By the sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Let each of these $= k$. Then

$$k \sin A = a, \quad k \sin B = b, \quad k \sin C = c.$$

Multiplying (i) throughout by k and substituting, we have

$$a = b \cos C + c \cos B.$$

This may be written $c \cos B = a - b \cos C$.

Also, by the sine rule, $c \sin B = b \sin C$.

$$\therefore \quad c^2 (\cos^2 B + \sin^2 B) = (a - b \cos C)^2 + b^2 \sin^2 C$$
$$= a^2 - 2ab \cos C + b^2 \cos^2 C + b^2 \sin^2 C;$$
$$\therefore \quad c^2 = a^2 - 2ab \cos C + b^2.$$

The "sine" and "cosine rules"* and the relation $A + B + C = 180°$ are theoretically sufficient to solve any triangle of which a side and two other parts are given. But the formula $c^2 = a^2 + b^2 - 2ab \cos C$ is unsuited for logarithmic calculations, and for this reason, where it is required to solve triangles with the use of logarithmic tables, formula 105 is employed, together with other which we shall now obtain.

206. To prove that

$$\sin \frac{A - B}{2} = \frac{a - b}{c} \cos \frac{C}{2} \quad \text{..........................(107)}$$

and

$$\cos \frac{A - B}{2} = \frac{a + b}{c} \sin \frac{C}{2} \quad \text{..........................(108)}$$

Since $\dfrac{a}{c} = \dfrac{\sin A}{\sin C}$ and $\dfrac{b}{c} = \dfrac{\sin B}{\sin C}$;

$$\therefore \quad \frac{a - b}{c} = \frac{\sin A - \sin B}{\sin C} = \frac{2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C}$$
$$= \frac{2 \sin \frac{1}{2}C \sin \frac{1}{2}(A - B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C} = \frac{\sin \frac{1}{2}(A - B)}{\cos \frac{1}{2}C}.$$

* The names "sine rule," "cosine rule," and "tangent rule" are convenient for remembering, but they are not universally used; so it is well, as a rule, not to employ them in referring to formulae, unless the formulae are themselves stated.

The second formula may be proved similarly. Start by expressing $(a+b)/c$ as a ratio of sines, and simplify.

Dividing the first of these formulae by the second, the formula (105) of § 204 is obtained.

Formulae (107), (108) are hardly so much known as they ought to be. They are very useful in finding by logarithms the third side of a triangle when two sides and the included angle are given, as we shall see presently.

[As the formulae are a little difficult to remember, it will probably be better to remember the method of obtaining them.]

207. To express the sine, cosine, and tangent of half any angle of a triangle in terms of the sides.

We start with the formula

$$\cos A = \frac{b^2+c^2-a^2}{2bc}.$$

Now $\quad 2\sin^2\frac{A}{2} = 1-\cos A \quad$ and $\quad 2\cos^2\frac{A}{2} = 1+\cos A;$

$$\therefore \quad 2\sin^2\frac{A}{2} = \frac{2bc-b^2-c^2+a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc};$$

$$\therefore \quad \sin^2\frac{A}{2} = \frac{\{a-(b-c)\}\ \{a+(b-c)\}}{4bc}$$

$$= \frac{(a+c-b)\ (a+b-c)}{4bc} = \frac{\frac{1}{2}(a+c-b)\times\frac{1}{2}(a+b-c)}{bc}.$$

Now let s stand for the semi-sum of the three sides of the triangle, *i.e.* $\qquad s = \frac{1}{2}(a+b+c).$

Then $\quad \frac{1}{2}(a-b+c) = s-b \quad$ and $\quad \frac{1}{2}(a+b-c) = s-c.$

$$\therefore \quad \boldsymbol{\sin^2\frac{A}{2} = \frac{(s-b)\ (s-c)}{bc}}, \quad \text{or} \quad \boldsymbol{\sin\frac{A}{2} = \sqrt{\frac{(s-b)\ (s-c)}{bc}}} \qquad \textbf{.........(109)}$$

Again,

$$2\cos^2\frac{A}{2} = 1+\frac{b^2+c^2-a^2}{2bc} = \frac{2bc+b^2+c^2-a^2}{2bc};$$

$$\therefore \quad \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{\{(b+c)-a\}\,\{(b+c)+a\}}{4bc}$$

$$= \frac{\frac{1}{2}(a+b+c)\times\frac{1}{2}(b+c-a)}{bc}.$$

Introducing s as before, we have

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}, \quad \text{or} \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \quad \ldots\ldots(110)$$

Dividing (109) by (110), we have

$$\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)}, \text{ or } \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad (111)$$

The last result may also be obtained independently without previously proving (109), (110), thus—

$$\tan^2\frac{A}{2} = \frac{1-\cos A}{1+\cos A} = \frac{2bc-(b^2+c^2-a^2)}{2bc+(b^2+c^2-a^2)},$$

and so on.

Illustrative Exercise.

Obtain $\tan^2 \frac{1}{2}B$ in this way.

Note.—The *Principle of Symmetry* will be found useful in writing down these formulae correctly. Thus $(s-a)(s-b)/ab$ is symmetrical in a and b but not in c, and therefore by the sine rule it represents a function which is unaltered by interchanging the angles A and B, and this must be a function of C, not of A or B. From the formulae above this function is $\sin^2 \frac{1}{2}C$. Again, if a function of one angle were written down as $s(s-a)/ab$, we should infer that it was incorrect, because any function of an angle must be symmetrical with respect to the two sides containing that angle.

208. To express sin A in terms of the sides.

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}$$

$$= \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}} = \frac{2}{bc}S,$$

where $$S = \sqrt{\{s(s-a)(s-b)(s-c)\}} \quad \ldots\ldots\ldots\ldots\ldots(112)$$

This result may be obtained without assuming the values of $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$, thus—

$$\sin^2 A = 1-\cos^2 A = (1-\cos A)(1+\cos A)$$

$$= \frac{2bc-(b^2+c^2-a^2)}{2bc}\cdot\frac{2bc+(b^2+c^2-a^2)}{2bc}, \text{ etc.}$$

COR.—Hence we have an independent verification of the sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

each member being equal to $\frac{2S}{abc}$.

We shall now prove that S represents the area of the triangle.

209. To find an expression for the area of a triangle.

Let $\triangle$ be the area of the triangle **ABC**. Then shall

$\triangle = \frac{1}{2}$ product of two sides into the sine of the included angle ....(113)

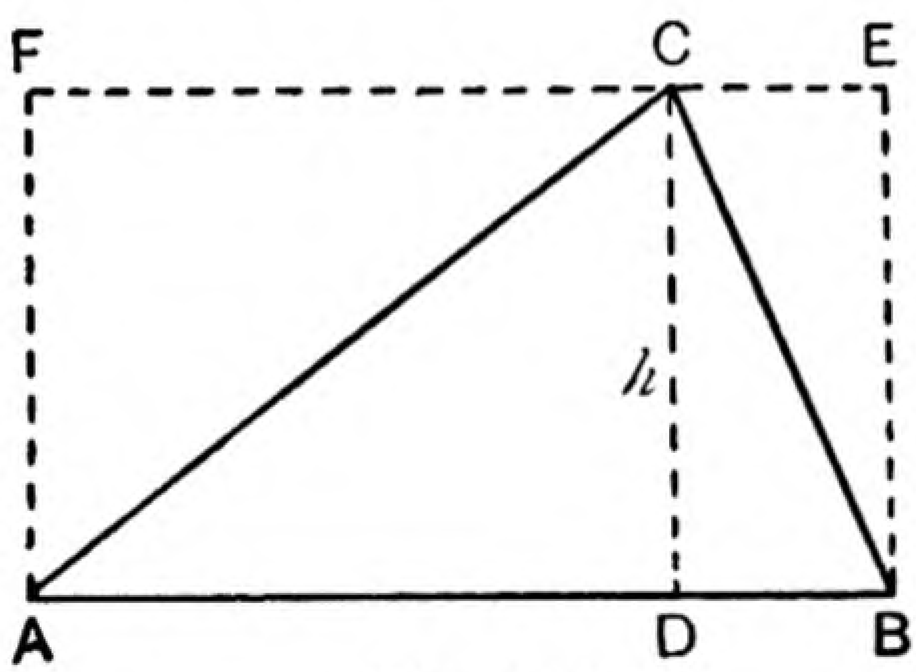


Fig. 111.

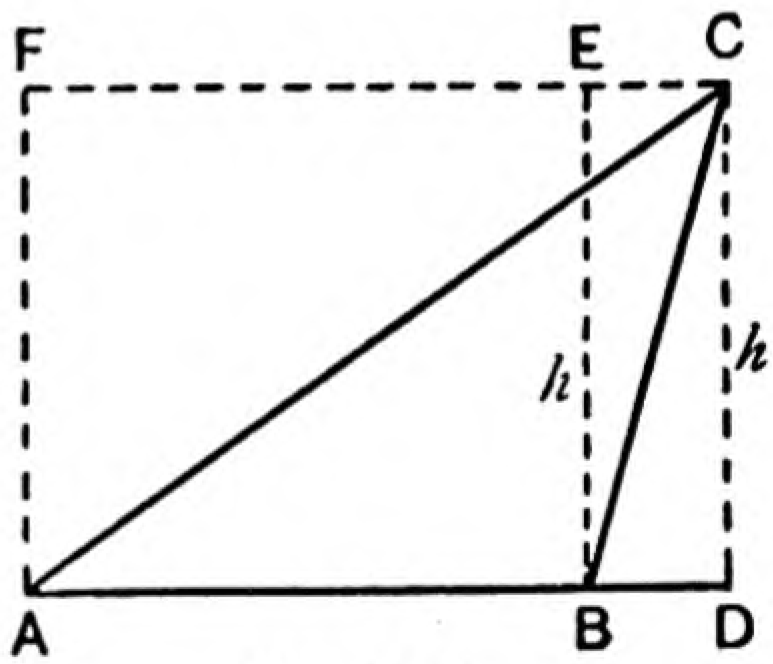


Fig. 112.

Draw the perpendicular **CD**, and construct the rectangle **ABEF** on the base **AB** with the same altitude as the triangle. Then, if **DC** $= h$, **BE** $= h$, and, by Euc. I. 41,

$$\triangle = \tfrac{1}{2} \text{ [rectangular] parallelogram on same base}$$
$$= \tfrac{1}{2} \text{ rect. } \mathbf{AB}.\mathbf{BE} = \tfrac{1}{2}\, ch.$$

But $$h = b \sin A;$$

$$\therefore \quad \triangle = \tfrac{1}{2}\, bc \sin A \quad \text{..................(113)}$$

210. To express the area in terms of the sides.

Since (by § 208) $\sin A = \frac{2S}{bc}$,

$$\therefore \quad \triangle = \tfrac{1}{2}\, bc\, \frac{2S}{bc} = S;$$

or, remembering the expression for S, we have

$$\textbf{area of triangle} = \sqrt{\{s(s-a)(s-b)(s-c)\}} \text{.........} \textbf{(114)}$$

If the lengths are measured in feet, this gives the area in square feet, and so on.

Ex. The sides of a triangle are 13, 14, and 15 metres long. Find its area in square metres.

Here $s = \frac{1}{2}\{13+14+15\} = 21;$

$$\therefore \quad s-a = 8, \quad s-b = 7, \quad s-c = 6;$$

$$\therefore \quad \Delta = \sqrt{\{21\times 8\times 7\times 6\}} = \sqrt{\{3.7\times 2^3\times 7\times 3.2\}}$$

$$= \sqrt{\{3^2\times 7^2\times 2^4\}} = 3\times 7\times 2^2 = 84.$$

Hence the area is 84 square metres.

EXAMPLES XVIII.

Tables of logarithms are to be used with Exx. 56–63.

1. Prove that, in any triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

2. Prove that, in any triangle, $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$.

3. Find the cosine of the largest angle of the triangle whose sides are 8, 11, and 14 ft. respectively.

4. Prove, without using logarithms, that the smallest angle of the triangle whose sides are 10, 17, 21 is less than 30°.

5. Prove, without using tables, that in a triangle whose sides are 3, 4, and $\sqrt{38}$ ft. in length, the largest angle is greater than 120°.

6. In the triangle whose sides are 7, 12, and 14 ft. in length respectively, find whether the least angle is greater or less than 30°.

7. In a triangle **ABC**, $a = \sqrt{5}$, $b = 2$, $c = \sqrt{3}$, prove that

$$8 \cos A \cos C = 3 \cos B.$$

8. In any triangle, show that $c = a \cos B + b \cos A$, and hence show that

$$\sin (A+B) = \sin A \cos B + \sin B \cos A.$$

9. Show that $c^2 = (a+b)^2 \sin^2 \dfrac{C}{2} + (a-b)^2 \cos^2 \dfrac{C}{2}$.

10. If **M** be the middle point of the base **BC** of a triangle **ABC**, and **D**, **H** the points where the bisector of the vertical angle and the perpendicular from the vertex respectively meet the base, prove that **MD** is to **MH** in the ratio a^2 to $(b+c)^2$.

11. If, in any triangle, the angle $A = 60°$, prove that
$$(a+b+c)(b+c-a) = 3bc.$$

12. In a triangle **ABC**, in which $a+b = 2c$, prove that
$$a \cos B - b \cos A = 2a - 2b.$$

13. Prove that, in any triangle, $a = b \cos C + c \cos B$, and from this and the corresponding formulae deduce
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

14. Between what limits is the tabular logarithmic secant of every angle contained?

If in a triangle the values of a, b, $\angle A$ are given, how ought the figure to be drawn?

Draw the figure in the following cases, if possible:—
$$\text{(i) } A < 90°,\ b \sin A < a < b; \quad \text{(ii) } A > 90°,\ a < b.$$

15. Prove that, in any triangle, $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s.(s-a)}}$.

16. Find the sine of half the smallest angle of the triangle whose sides are 5, 6, and 7 units long.

17. In a triangle, given $a = 35$, $b = 52$, $c = 63$, find
$$\tan \frac{A}{2}, \quad \tan \frac{B}{2}.$$

18. In a triangle, given $a = 25$, $b = 52$, $c = 63$, find $\tan \frac{C}{2}$.

19. In any triangle **ABC**, prove that $a \sin^2 \frac{B}{2} + b \sin^2 \frac{A}{2} = s - c$.

20. Prove that, in any triangle, $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$.

21. If **ABC** be a triangle having a right angle at **C**, show that
$$a\,(1 + \tan \tfrac{1}{2}B) = (b+c)\,(1 - \tan \tfrac{1}{2}B).$$

22. Prove the formula for the sine of an angle of a triangle in terms of the sides of the triangle.

23. In a triangle, given $a = 18$, $b = 24$, $c = 30$, find $\sin A$, $\sin B$, $\sin C$.

24. In a triangle, given $a = 13$, $b = 14$, $c = 15$, find $\sin A$, $\sin B$, $\sin C$.

25. In a triangle, given $a = 125$, $b = 123$, $c = 62$, find $\sin A$, $\sin B$, $\sin C$.

26. Find the areas of the following triangles, having given

(i) $a = 35$, $b = 84$, $c = 91$;
(ii) $a = 114$, $b = 101$, $c = 25$;
(iii) $a = 18$, $b = 24$, $c = 30$;
(iv) $a = 13$, $b = 14$, $c = 15$;
(v) $a = 25$, $b = 52$, $c = 63$.

27. **ABC** and **A′B′C′** are two triangles, which have the angles C and C' equal and coinciding with one another, and **A, A′, C** in one line; when **A′B** and **AB′** are drawn they are parallel; show that

$$c^2 : c'^2 :: \sin A' \sin B' : \sin A \sin B.$$

28. If the vertex **A** of a triangle **ABC** be joined to any point **D** in the base, show that $\mathbf{BC} \cot \mathbf{ADC} = \mathbf{DC} \cot B - \mathbf{BD} \cot C$.

29. If the tangents of the angles of a triangle are in arithmetical progression, show that the squares of the sides are in the ratios

$$x^2(x^2+9) : (3+x^2)^2 : 9(1+x^2),$$

where x is the least or greatest tangent.

30. Prove, geometrically, that the sum of two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.

31. **A** and **B** are two stations 1,000 ft. apart; **P** and **Q** are two stations in the same plane as **AB**, and on the same side of it; the angles **PAB, PBA, QAB,** and **QBA** are 75°, 30°, 45°, and 90° respectively; find the distance of **P** from **Q**, and how far each of them is from **A** and **B**.

32. Having measured a base **AB** and the angles **ABC, BAC,** where **C** is a distant object, and these angles are very nearly right angles, prove that the distance of **C** from **A** or **B** is approximately

$$\mathbf{AB} \operatorname{cosec} (180° - \mathbf{ABC} - \mathbf{BAC}).$$

33. Given area of a triangle and two of its sides, show how to find the angles and third side.

34. Find the area and the trigonometrical ratios of the angles of a triangle whose sides are 15, 36, and 39 ft.

35. If the area, the perimeter, and one of the angles of a triangle are given, show how to find the sides.

36. A triangle is on the same base as a parallelogram, which has an area and perimeter double those of the triangle; show that the cosecant of an angle of the parallelogram equals the sum of the cosecants of the angles at the base of the triangle.

37. Prove that, in any triangle, 4 times the area $= b^2 \sin 2C + c^2 \sin 2B$, and interpret the result geometrically.

38. In any triangle, show that

$$a^2b^2c^2(\sin 2A + \sin 2B + \sin 2C) = 32S^3,$$

where S denotes the area of the triangle.

39. If a triangle **ABC** be divided into two right-angled triangles by a line drawn from **A** at right angles to **BC**, show that twice the difference of the areas of the right-angled triangles is $= bc \sin(B-C)$, and hence show that

$$b^2 \sin 2C - c^2 \sin 2B = 2bc \sin(B-C).$$

40. The lengths of the sides of a triangle are in arithmetical progression, and its area is three-fifths of that of an equilateral triangle of the same perimeter; find the greatest angle of the triangle. If the perimeter is 300 ft., what are the lengths of the sides?

41. An equilateral triangle **ABC**, whose area is denoted by $\triangle$, is divided into two triangles by a line drawn through **A**; the perpendicular distances from this line of **B** and **C** are p and q respectively; show that

$$p^2+pq+q^2=\triangle\sqrt{3}.$$

42. If the angle at **C** be a right angle, show that

$$\frac{a^2\sin A}{1+\cos A}-\frac{b^2\cos A}{1+\sin A}=\frac{c^2\sin(A-B)}{\sin A+\cos A}.$$

43. Find an expression for the vertical angle of a triangle, given the altitude, the base, and the difference of the base angles. Illustrate by a diagram the two solutions of the problem.

44. Show that

$$abc\,(1-2\cos A\cos B\cos C)$$
$$=a^3\cos B\cos C+b^3\cos C\cos A+c^3\cos A\cos B.$$

45. Show that, in any triangle,

$$(2a-b-c)\sin\frac{B-C}{2}\sin\frac{A}{2}+(2b-c-a)\sin\frac{C-A}{2}\sin\frac{B}{2}$$
$$+(2c-a-b)\sin\frac{A-B}{2}\sin\frac{C}{2}=0.$$

46. If, in a triangle,

$$(a+b)\,c\cos\frac{B}{2}=(a+c)\,b\cos\frac{C}{2},\text{ then } b=c.$$

47. Show that
$$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}=\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

48. Show that

$$\frac{\sin(B-C)}{\sin A}+\frac{\sin(C-A)}{\sin B}+\frac{\sin(A-B)}{\sin C}$$
$$=\frac{-4\sin(B-C)\sin(C-A)\sin(A-B)}{\sin 2A+\sin 2B+\sin 2C}.$$

49. Show that, in any triangle **ABC**,

(i) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;

(ii) $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$.

(iii) $\cos A+\cos B+\cos C>1$.

50. Prove that

$$\sin A\cos^2 A\sin(B-C)+\sin B\cos^2 B\sin(C-A)$$
$$+\sin C\cos^2 C\sin(A-B)=0.$$

51. If A, B, C be angles of a triangle, prove that

$$\tan \tfrac{1}{2}A+\tan \tfrac{1}{2}B+\tan \tfrac{1}{2}C = 4\,\frac{1+\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C}{\sin A+\sin B+\sin C}.$$

52. If an equilateral triangle be described, having its angular points on three parallel straight lines, the distances of the middle one from the two outside ones being a and c respectively, prove that the side of the triangle is equal to $2\sqrt{\dfrac{a^2+ac+c^2}{3}}$.

53. Prove that

$$\frac{a}{\cos A}\left(\frac{\cos B}{b}+\frac{\cos C}{c}\right)+\frac{b}{\cos B}\left(\frac{\cos C}{c}+\frac{\cos A}{a}\right)+\frac{c}{\cos C}\left(\frac{\cos A}{a}+\frac{\cos B}{b}\right)$$
$$= \sec A \sec B \sec C-2.$$

54. Prove that

$$\frac{\cot B+\cot C}{\cot \frac{1}{2}B+\cot \frac{1}{2}C}+\frac{\cot C+\cot A}{\cot \frac{1}{2}C+\cot \frac{1}{2}A}+\frac{\cot A+\cot B}{\cot \frac{1}{2}A+\cot \frac{1}{2}B}=1.$$

55. Show that $(s-a)^2 \sin A+(s-b)^2 \sin B+(s-c)^2 \sin C$

$$= 2S\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right).$$

56. The diagonals of a parallelogram make an angle of 35° with one another, and are severally 117·72 and 157·41 ft. long. What is the area of the parallelogram ?

57. Find the cosine of the largest angle of the triangle whose sides are 8, 11, and 14 ft. long, and find the angle itself.

58. **ABCD** is a quadrilateral whose diagonals **AC** and **BD** intersect in **E**. $\angle$**AEB** = 105° 20′, **AC** = 343·64 ft., **BD** = 673·75 ft. Find the area of the quadrilateral.

59. Calculate to the nearest minute the smallest angle of the triangle whose sides are 8, 9, 13.

60. The diagonal of a rectangle is 638·64 ft., and makes an angle of 106° 9′ with the other diagonal; calculate the area of the rectangle.

61. The sides of a triangle are 7, 8, and 9 units long; find the sine of half the smallest angle and the angle itself to the nearest minute.

62. Given $a = 6$, $b = 5$, $c = 10$, find $\cos C$ and from it find C.

63. Calculate to the nearest minute the greatest angle of the triangle whose sides are 35, 40, and 45 ft. long.

CHAPTER XIX.

LOGARITHMIC SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

211. We shall now apply the formulae of the last chapter to solving oblique-angled triangles with the aid of logarithmic tables.

In the solution of numerical problems we occasionally require to work with functions of obtuse angles, although the tables only refer to angles between 0° and 90°. In dealing with the *sines* of obtuse angles and their logarithms, the formula $\sin A = \sin(180° - A)$ shows that we merely have to take the sine of the supplement of the angle; similarly for the cosecant.

E.g.
$$\begin{aligned} L\sin 156°\,44' &= L\sin(180° - 156°\,44') \\ &= L\sin 23°\,16' = 9{\cdot}59661. \end{aligned}$$

The cosines and tangents of obtuse angles are negative, and the logarithms of negative quantities, like the square root of a negative quantity, are imaginary; but it is always possible to transform formulae in such a way as to work with logarithms of positive quantities alone, only the *numerical values* of negative quantities being calculated by logarithms, and their signs being allowed for separately. Such cases, however, will be found rarely to occur.

The various cases occurring in the solution of triangles will be taken in the following order:—

Case I.—When two angles and a side are given

Case II.—When two sides and the angle opposite one are given.

Case III.—When two sides and the included angle are given.

Case IV.—When three sides are given.

212. Case I.—Given two angles and a side, *to solve the triangle.*

First find the third angle from the relation

$$A+B+C = 180°.$$

The remaining sides can now be found by the sine rule. Thus, if a be the given side,

$$b = a\frac{\sin B}{\sin A}, \quad c = a\frac{\sin C}{\sin A}.$$

These relations are adapted to logarithmic computation.

Taking logarithms, the *best* form to use, if a book of tables is at hand, is

$$\log b = \log a + L \operatorname{cosec} A + L \sin B - 20,$$
$$\log c = \log a + L \operatorname{cosec} A + L \sin C - 20,$$

because in these we only have to *add* three logs together. If we worked with $L \sin A$ instead of $L \operatorname{cosec} A$, we should have to subtract it from the sum of the other two logs.

Ex. Given $c = 5280$ ft., $C = 30°$, $A = 128°$, find a, b.

The third angle B is given by $B = 180°-128°-30° = 22°$, and to find the remaining sides we have

$$a = 5280\frac{\sin 128°}{\sin 30°}, \qquad b = 5280\frac{\sin 22°}{\sin 30°}.$$

Remembering

$$\sin 128° = \sin(180°-128°) = \sin 52°,$$

we have

$$\log a = \log 5280 + L \sin 52° - L \sin 30°,$$

and

$$\log b = \log 5280 + L \sin 22° - L \sin 30°,$$

$$\begin{array}{rlrl}
\log 5280 = & 3{\cdot}72263, & = & 3{\cdot}72263, \\
L \sin 52° = & 9{\cdot}89653, & L \sin 22° = & 9{\cdot}57358, \\
\hline
 & 13{\cdot}61916, & & 13{\cdot}29621, \\
L \sin 30° = & 9{\cdot}69897, & L \sin 30° = & 9{\cdot}69897, \\
\hline
 & 3{\cdot}92019, & & 3{\cdot}59724,
\end{array}$$

$$\therefore \quad a = \text{antilog } 3{\cdot}92019, \qquad b = \text{antilog } 3{\cdot}59724,$$
$$= 8321{\cdot}2 \text{ ft.}, \qquad = 3955{\cdot}9 \text{ ft.},$$

213. Case II.—Given two sides and the angle opposite one of them, *to solve the triangle.*

First find the angle opposite the other of the given sides by the sine rule. Thus, if a, b, A are the given parts, then

$$\sin B = \frac{b}{a} \sin A \text{ ....(1)}$$

This gives, when translated into logarithmic form,

$$L \sin B = \log b + L \sin A - \log a.$$

The value **or values** of B being thus found, the third angle is thus obtained from the relation—

$$A+B+C = 180°, \quad \text{or} \quad C = 180° - (A+B) \text{ ....(2)}$$

and the third side c is then found from the sine rule

$$c = a\frac{\sin C}{\sin A} = b\frac{\sin C}{\sin B} \text{ ....(3)}$$

the only new logarithm required being $L \sin C$.

In equation (1) the angle B is determined from the value of its sine, and since trigonometric tables are only calculated for angles between 0° and 90°, these tables will give the *acute* angle satisfying the relation in question. If this acute angle be B, then, since

$$\sin (180° - B) = \sin B,$$

$180° - B$ is *another* angle satisfying the equation, and is *obtuse.*

Either of these two angles will be inadmissible as a solution of the problem if, when added to the given angle A, the sum is greater than 180°; for then C would be negative by (2). But if both of the angles give a positive value for C, both solutions are admissible, and there are *two* triangles which have the given parts, just in the same way that in algebra a quadratic equation has *two* roots. In such cases the solution is said to be **ambiguous.**

The conditions under which this occurs will be considered fully in § 216. But in any problem where numerical data are given all that is necessary is to write down the two values of B and ascertain by substitution if they separately satisfy the condition $A+B<180°$.

Ex. If $A = 32° 17'$, $a = 1952{\cdot}1$, $b = 356{\cdot}2$, find B and C.

By the sine rule $$\frac{\sin B}{b} = \frac{\sin A}{a},$$

which, when adapted to logarithms, becomes

$$L \sin B = \log b + L \sin A - \log a,$$

$$\therefore \quad L \sin B = \log 356{\cdot}2 + L \sin 32° 17' - \log 1952{\cdot}1$$

$$= 2{\cdot}55169 - 3{\cdot}29049$$

$$+ \underline{9{\cdot}72764}$$

$$= 12{\cdot}27933 - 3{\cdot}29049$$

$$= 8{\cdot}98884,$$

$$\therefore \quad B = 5° 36', \text{ or its supplement } 174° 24'.$$

This last is impossible, for then B and A would together exceed two right angles.

$$\therefore \quad B = 5° 36',$$

$$\text{and } C = 180° - A - B$$

$$= 180° - 32° 17' - 5° 36'$$

$$= 142° 7'.$$

NOTE.—It is always advisable to start by writing down the fundamental formula and then deduce the corresponding logarithmic form from it.

214. Case III.—Given two sides and the included angle, *to solve the triangle.*

FIRST METHOD.—*If logarithms are not used* the simplest plan is to find the third side by the "cosine" rule. One of the remaining angles may then be found by the "sine" rule, and the third angle by Euclid I. 32.

Thus, given b, c, A, we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and $$\sin B = \frac{b \sin A}{a}, \quad C = 180° - A - B.$$

Ex. If $a = 7$ ft., $b = 9$ ft., $C = 120°$, find c.

Here $$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= 49 + 81 - 2.7.9\,(-\tfrac{1}{2}) = 49 + 81 + 63 = 193;$$

$$\therefore \quad c = 13{\cdot}89 \text{ ft.}$$

SECOND METHOD.—*When logarithmic tables have to be used.*

First find the other two angles by means of Napier's formula (the "tangent rule").

Thus if a, b, C are given,

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

In logarithmic form

$$L\tan\tfrac{1}{2}(A-B)=L\cot\tfrac{1}{2}C+\log(a-b)-\log(a+b).$$

When $\frac{A-B}{2}$ is found, **it should not be doubled.**

$$\frac{A+B}{2}\text{ is known, }=90^\circ-\frac{C}{2}.$$

Add, we obtain A; *subtract*, we obtain B.

The third side c can now be found by either of the formulae of § 206, viz.—

$$\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}\text{ or }\cos\frac{A-B}{2}=\frac{a+b}{c}\sin\frac{C}{2}.$$

The second is the better formula to use, because it does not fail when a is nearly $=b$. The best logarithmic form to take is

$$\log c=\log(a+b)+L\sin\frac{C}{2}+L\sec\frac{A-B}{2}-20.$$

Another way of calculating the third side c when the angles have been found is by using the sine rule—

$$\frac{c}{\sin C}=\frac{a}{\sin A}=\frac{b}{\sin B}.$$

This is the method usually given in textbooks, but although it looks simpler at first sight, it requires an entirely new set of logarithms to be taken from the tables, and the work is therefore far more laborious.

Ex. 1. Given

$$A=47^\circ\,18',\quad b=2516,\quad c=1472,$$

find B and C.

Here $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2},$$

$$\therefore\ L\tan\frac{B-C}{2} = L\cot\frac{A}{2}+\log(b-c)-\log(b+c)$$

$$= L\cot 23^\circ\ 39'+\log 1044-\log 3988$$

$$= 10{\cdot}35860-3{\cdot}60076$$

$$+\ \ 3{\cdot}01870$$

$$= 13{\cdot}37730-3{\cdot}60076$$

$$= 9{\cdot}77654,$$

$$\therefore\ \frac{B-C}{2} = 30^\circ\ 52'.$$

Also
$$\frac{B+C}{2} = 90^\circ - \frac{A}{2}$$

$$= 66^\circ\ 21',$$

$$\therefore\ \text{adding, } B = 97^\circ\ 13',$$

$$\text{subtracting, } C = 35^\circ\ 29'.$$

Ex. 2. Given

$$b = 5038 \text{ metres},\quad c = 6840 \text{ metres},\quad A = 94^\circ\ 16',$$

find a.

Since A is the given angle, and $c>b$, we use the formula

$$\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$$

or
$$L\tan\frac{C-B}{2} = L\cot\frac{A}{2}+\log(c-b)-\log(c+b)$$

$$= L\cot 47^\circ\ 8'+\log 1802-\log 11878$$

$$= 9{\cdot}96763-4{\cdot}07475$$

$$+\ \ 3{\cdot}25576$$

$$= 13{\cdot}22339-4{\cdot}07475$$

$$= 9{\cdot}14864.$$

$$\therefore\ \frac{C-B}{2} = 8^\circ\ 1' \quad\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

and
$$\frac{C+B}{2} = 90^\circ - \frac{A}{2}$$

$$= 42^\circ\ 52',$$

$$\therefore\ \text{adding, } C = 50^\circ\ 53'.$$

Now $$\frac{a}{c}=\frac{\sin A}{\sin C},$$

$$\begin{aligned}\therefore\ \log a &= \log c+L\sin A-L\sin C\\ &= \log 6840+L\sin(180^\circ-94^\circ 16')-L\sin 50^\circ 53'\\ &= \begin{array}{l}3{\cdot}83506-9{\cdot}88979\\ 9{\cdot}99880\\ \hline\end{array}\\ &= 13{\cdot}83386-9{\cdot}88979\\ &= 3{\cdot}94407,\\ \therefore\ a &= \text{antilog } 3{\cdot}94407\\ &= 8791{\cdot}6 \text{ metres.}\end{aligned}$$

The following is an alternative method of procedure from the point marked (a), reducing the use of the tables:—

$$\sin\frac{C-B}{2}=\frac{c-b}{a}\cos\frac{A}{2},$$

$$\begin{aligned}\therefore\ \log a &= \log(c-b)+L\cos\frac{A}{2}-L\sin\frac{C-B}{2}\\ &= \log 1802+L\cos 47^\circ 8'-L\sin 8^\circ 1'\\ &= \begin{array}{l}3{\cdot}25576-9{\cdot}14445\\ +\ 9{\cdot}83269\\ \hline\end{array}\\ &= 13{\cdot}08845-9{\cdot}14445\\ &= 3{\cdot}94400\\ \therefore\ a &= \text{antilog } 3{\cdot}94400\\ &= 8790{\cdot}2 \text{ metres,}\end{aligned}$$

a result differing slightly from the former result.

215. Case IV.—Given three sides, *to solve the triangle.*

First Method.—If the lengths of the sides are numbers of one or two figures only, the cosines of two of the angles may be found by the cosine rule, the third angle being given by Euclid I. 32.

Thus $$\cos A=\frac{b^2+c^2-a^2}{2bc},\quad \cos B=\frac{c^2+a^2-b^2}{2ca},$$

and $$C=180^\circ-(A+B).$$

[See § 202, Ex., for example.]

SECOND METHOD.—*When it is necessary to use logarithms* to shorten the work, the formulae for the sine, cosine, or tangent of the semi-angles may be used, namely,

$$\sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}, \quad \cos\frac{A}{2} = \sqrt{\left\{\frac{s.(s-a)}{bc}\right\}},$$

or,
$$\tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s.(s-a)}\right\}},$$

where $s = \frac{1}{2}(a+b+c) =$ *half* sum of sides.

Where only *one* angle of the triangle is required, either of these three formulae may be used equally well.

But where *all* the angles are required the formula for the *tangent* is the best to use; thus

$$\tan\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s.(s-a)}\right\}}.$$

$$\tan\frac{B}{2} = \sqrt{\left\{\frac{(s-c)(s-a)}{s.(s-b)}\right\}},$$

and
$$C = 180^\circ-(A+B).$$

For the two first of these, when written in logarithmic form, both involve the same four logarithms, namely, those of s, $s-a$, $s-b$, $s-c$; whereas, if we were to determine two of the semi-angles from their sines or cosines, six logarithms instead of four would have to be taken from the tables, only two being common to both formulae. The most convenient logarithmic forms to make the computation as short as possible are—

$L \tan\frac{1}{2}A = [10+\frac{1}{2}\{\log(s-a)+\log(s-b)+\log(s-c)-\log s\}]-\log(s-a)$,
$L \tan\frac{1}{2}B = [$ same expression $]-\log(s-b)$,
$L \tan\frac{1}{2}C = [$ same expression $]-\log(s-c)$.

Ex. 1. The sides of a triangle, measured in feet, are—

$$a = 3294, \quad b = 4516, \quad c = 1542;$$

to find A, B, C.

3294	4676	4676	4676
4516	3294	4516	1542
1542			
	$s-a = 1382$	$s-b = 160$	$s-c = 3134$
2) 9352			
$s = 4676$			

Now
$$\begin{aligned} \log(s-a) &= \log 1382 = 3{\cdot}14050 \\ \log(s-b) &= \log\ 160 = 2{\cdot}20412 \\ \log(s-c) &= \log 3134 = 3{\cdot}49609 \\ 20-\log s = 20-\log 4676 &= 20-3{\cdot}66988 = 16{\cdot}33012 \end{aligned}$$

(by addition) $2\,)\,25{\cdot}17083$

$$\therefore\ 10+\tfrac{1}{2}\{\log(s-a)+\log(s-b)+\log(s-c)-\log s\} = 12{\cdot}58542$$
$$\text{subtract } \log(s-a) = 3{\cdot}14050$$
$$\therefore\ L\tan\tfrac{1}{2}A = 9{\cdot}44492$$
$$\therefore\ \tfrac{1}{2}A = 15^\circ\,34',\text{ or } A = 31^\circ\,8'.$$

Again,
$$10+\tfrac{1}{2}\{\log(s-a)+\log(s-b)+\log(s-c)-\log s\} = 12{\cdot}58542$$
$$\text{subtract } \log(s-b) = 2{\cdot}20412$$
$$\therefore\ L\tan\tfrac{1}{2}B = 10{\cdot}38130$$
$$\therefore\ \tfrac{1}{2}B = 67^\circ\,26',\text{ or } B = 134^\circ\,52'.$$

Lastly, $$C = 180^\circ-31^\circ\,8'-134^\circ\,52' = 14^\circ\,0'.$$

Ex. 2. The sides of a triangle are $a = 7$, $b = 8$, $c = 9$; find A.
$$L\tan\tfrac{1}{2}A-10 = \tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}.$$

Here $$s = \tfrac{1}{2}(7+8+9) = 12;$$
$$\begin{aligned} \therefore\ L\tan\tfrac{1}{2}A &= 10+\tfrac{1}{2}\{\log 4+\log 3-\log 12-\log 5\} \\ &= 10-\tfrac{1}{2}\log 5 = 10-\tfrac{1}{2}(1-\log 2) \\ &= 10-\tfrac{1}{2}(\cdot 69897) = 10-\cdot 34949 \\ &= 9{\cdot}65051. \end{aligned}$$
$$\therefore\ \tfrac{1}{2}A = 24^\circ\,6' \text{ or } A = 48^\circ\,12'.$$

216. The Ambiguous Case.—The conditions under which an ambiguity occurs in solving a triangle when the given parts are two sides and the angle opposite one of them can be best discussed geometrically by considering the corresponding geometrical problem:—

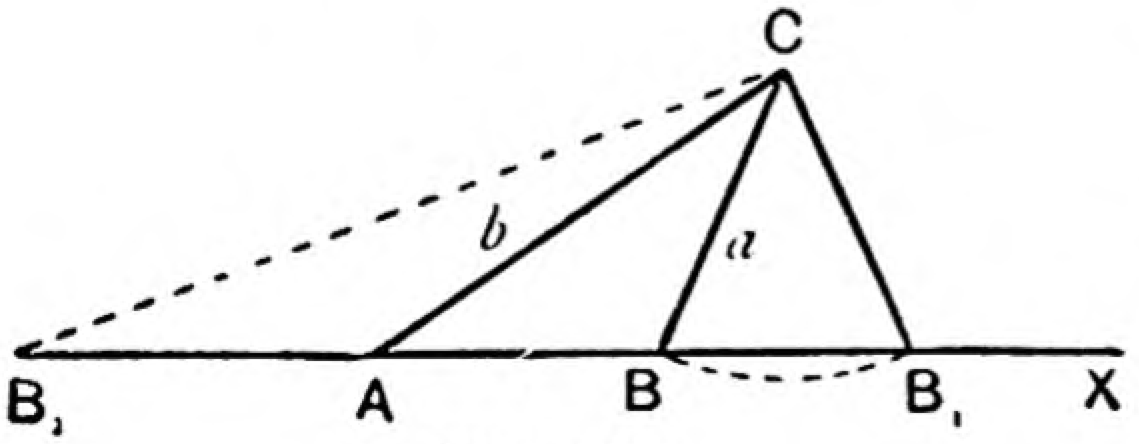


Fig. 113.

Given two sides of a triangle and the angle opposite one of them, to construct the triangle.

Let a, b, A be the given parts. Make $\angle$ **XAC** $= A$, and cut off **AC** $= b$. With centre **C** and radius a describe a circle. If this circle cuts **AX**,

let the points of section be **B** and $\mathbf{B}_1$. Then, if **B** falls on the same side of **A** as **X**, the triangle **ABC** will have the given parts, but, if **B** falls on the opposite side of **A** as at $\mathbf{B}_2$, the triangle $\mathbf{AB_2C}$ will not satisfy the given conditions, the angle $\mathbf{B_2AC}$ being the supplement of A. Similarly for $\mathbf{B}_1$. Hence the number of solutions of the problem will be the number of points in which the circle cuts the line **AX** on the side of **A** towards **X**.

The student will have no difficulty now in verifying the following results, *which are left to be proved as an illustrative exercise:—*

If $a > b$, there will be *one* triangle.

If $a = b$, there will be *one* triangle if A be *acute*, *no* triangle if A be *obtuse*.

If $a < b$, but $a > b \sin A$ (the length of the perpendicular from **C** on **AB**), there will be *two* triangles if A be *acute*, *no* triangles if A be *obtuse*.

If $a = b \sin A$, the two triangles will coincide, and will be right-angled at **B**; hence there will be one triangle.

If $a < b \sin A$, there will be *no* triangle satisfying the given conditions, for the circle will not cut **AX**.

Hence, for an ambiguity, A must be acute, and a lie between b and $b \sin A$ in magnitude.

EXAMPLES XIX.

1. Two angles of a triangle being $18° \ 20'$ and $11° \ 40'$, and the longest side 10,000 ft., find the length of the shortest side.

2. If $A = 55°$, $B = 65°$, $c = 270$, find a.

3. If $a = 1020$, $B = 107° \ 18'$, $C = 27° \ 10'$, find b.

4. Given $A = 53° \ 24'$, $B = 66° \ 27'$, $c = 338{\cdot}65$ ft., find the length of a.

5. Two adjacent sides of a triangle are 55 and 40, and the opposite angle to the greater side is $54° \ 10'$: find the angle opposite the less.

6. Two sides of a triangle are 21154 ft. and 17308 ft., respectively, and they include an angle of $53° \ 42'$: find the other angles.

7. Two sides of a triangle are 535 ft. and 465 ft., respectively, and the angle between them is $51° \ 20'$: find the other angles.

8. The sides of a triangle being 237·09 ft. and 130·96 ft., and the included angle $57° \ 58'$, find the remaining angles.

9. The sides 2265·4 and 1779 being given, and the included angle $58° \ 17'$, find the remaining angles of the triangle.

10. In a triangle, given $a = 456{\cdot}12$, $b = 296{\cdot}86$, $C = 74° \ 20'$, find A and B.

11. If $\tan \phi = \dfrac{a-b}{a+b} \cot \tfrac{1}{2}C$, prove that $\phi = \tfrac{1}{2}(A-B)$ and that

$$c = \frac{(a+b) \sin \tfrac{1}{2}C}{\cos \phi}.$$

12. In the last question, if the sides are 237 and 158, and the included angle $66° 40'$, find c from that formula.

13. If $a = 30$, $b = 10$, $C = 53°$, find c.

14. Given $a = 17$, $b = 6$, $C = 127° 40'$, find A and B.

15. Given, in a triangle, $a = 234$, $b = 129$, $C = 84° 24'$, find A and B.

16. Given $b = 72$, $c = 56$, $A = 70°$, find a.

17. Two sides of a triangle are 200 and 250 yd. respectively, and the included angle is 60°. Find the other angles.

18. Find the largest angle in the triangle whose sides are 8, 11, and 14 ft., respectively.

19. The sides of a triangle are 7, 8, 9: calculate the value of its smallest angle.

20. Find the greatest angle in a triangle whose sides are 5, 6, 7 yd., respectively.

21. In a triangle, if $a = 35225$, $b = 51327$, $c = 48268$, find A.

22. In the triangle **ABC**, $a = 97$, $b = 74$, $c = 90$; find A, B, C.

23. The lengths of two sides of a triangle are 5374·5 ft. and 1586·6 ft.; the angle opposite to the shorter side is $15° 11'$. Calculate the other two angles of the triangle, or of the triangles if there are two.

24. In a triangle **ABC**, given $A = 10°$, $a = 2308{\cdot}7$, $b = 7903{\cdot}2$, find the smaller value of c.

25. Find the greater value of c, given $A = 35° 36'$, $a = 1770$, $b = 2164{\cdot}5$.

26. In a plane triangle, prove that

$$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \tfrac{1}{2}A.$$

There is a plane quadrilateral **ABCD**, in which **AB** = 193 ft., ∠**BAC** = 37°, ∠**CAD** = 21°, ∠**ABD** = 59°, ∠**CBD** = 23°; find the length of **CD**.

27. The base **BC** of an isosceles triangle **ABC** is 1,300 ft. long, and the altitude is double that of an equilateral triangle on an equal base; the angles A, B, C are bisected by lines which meet at **D**. Find the angle **DAB**, and the number of square feet in the area of the triangle **DBC**.

28. What is meant by the ambiguous case in the solution of triangles? If two sides a and b and the angle A of the triangle **ABC** are given, under what circumstances will there be two values for the third side? Show that the difference of these values is

$$2\sqrt{a^2-b^2 \sin^2 A}.$$

Can a triangle be found for which $a = 118$, $b = 235$, $A = 31° 8'$?

29. Given that a side **BC** of a triangle is 450 ft. long, and that the perpendiculars drawn from **B** and **C** on the opposite sides are, respec-

tively, 400 ft. and 300 ft., show that there are two triangles which fulfil the conditions, and calculate the angles of each triangle.

Show that there are not more than two triangles which fulfil the given conditions.

Solve the triangles (30–36) of which the following elements are given:—

30.	$B = 26° 30'$,	$C = 47° 15'$,	$a = 1652$.
31.	$A = 89° 9'$,	$B = 54° 33'$,	$a = 15236$.
32.	$A = 60°$,	$b = 14$,	$c = 11$.
33.	$C = 37°$,	$a = 284$,	$b = 482$.
34.	$a = 60$,	$b = 160$,	$c = 180$.
35.	$a = 7853$,	$b = 6216$,	$A = 77° 35'$.
36.	$A = 48° 3'$,	$B = 40° 14'$,	$c = 376$.

37. Given $B = 29° 17'$, $C = 135°$, $a = 12300$, find c.

38. Given $B = 10°$, $C = 45°$, $a = 200$, find c.

39. Given $A = 25° 30'$, $b = 1285$, $c = 270$, find B and C.

40. Given $C = 44°$, $a = 43$, $b = 11$, find A and B.

41. Given $A = 40°$, $b = 131$, $c = 72$, find B and C.

42. Given $a = 32$, $b = 40$, $c = 66$, find the greatest angle, through $\cos \frac{C}{2}$.

43. Given $a = 131$, $b = 106$, $c = 75$, find A, through $\tan \frac{A}{2}$.

44. Given $a = 4$, $b = 5$, $c = 6$, find B, through $\cos \frac{B}{2}$.

45. Given $a = 27535$, $b = 18928$, $c = 30147$, find A, through $\tan \frac{A}{2}$.

46. Given $A = 41° 10'$, $a = 178$, $b = 145$, find B and C.

47. Given $A = 120°$, $a = 8$, $b = 7$, find B and C.

48. Given $C = 32° 15'$, $a = 468$, $c = 320$, find B.

49. Given $A = 72° 5'$, $a = 250$, $b = 240$, find the other angles.

Solve the following triangles (50–69):—

50.	$A = 36° 17'$,	$B = 47° 16'$,	$c = 1000$.
51.	$A = 19° 6'$,	$B = 18° 10'$,	$c = 280$.
52.	$B = 63° 7'$,	$C = 60° 5'$,	$c = 93$.
53.	$A = 100°$,	$B = 18°$,	$c = 1683$.
54.	$A = 14°$,	$B = 15°$,	$c = 36$.

55.	$A = 83° 6'$,	$a = 95$,	$b = 96$.
56.	$A = 15° 14'$,	$a = 183$,	$b = 200$.
57.	$A = 18° 36'$,	$a = 1896$,	$b = 1899$.
58.	$C = 17° 16'$,	$a = 12$,	$c = 96$.
59.	$B = 25°$,	$a = 17$,	$b = 83$.
60.	$A = 15° 28'$,	$b = 18$,	$c = 15$.
61.	$A = 27° 37'$,	$b = 25$,	$c = 18$.
62.	$A = 37° 2'$,	$b = 90$,	$c = 36$.
63.	$B = 22° 10'$,	$a = 18$,	$b = 29$.
64.	$C = 12° 14'$,	$a = 12365$,	$b = 12364$.
65.	$a = 12$,	$b = 21$,	$c = 30$.
66.	$a = 6$,	$b = 7$,	$c = 8$.
67.	$a = 80$,	$b = 90$,	$c = 100$.
68.	$a = 1262$,	$b = 1364$,	$c = 1672$.
69.	$a = 3672$,	$b = 7603$,	$c = 3998$.

CHAPTER XX.

APPLICATION OF TRIGONOMETRY TO LAND SURVEYING.

217. The most important practical use of the methods of solving triangles consists in their application to the determination of heights and distances in forming a trigonometric survey of a country. The following problem forms a convenient introduction to the subject:—

218. To find the distance of an inaccessible object from a given (accessible) place.

Let **C** be the object, **A** the given place of observation or "station." Take any other station **B** not lying in the straight line **AC.** Measure the distance **AB,** and observe the angles **CAB, CBA** subtended by **CB** at **A** and by **CA** at **B.** Then, knowing these two angles and the side **AB,** the triangle **ABC** may be solved, and the required distance **AC** determined from the formula

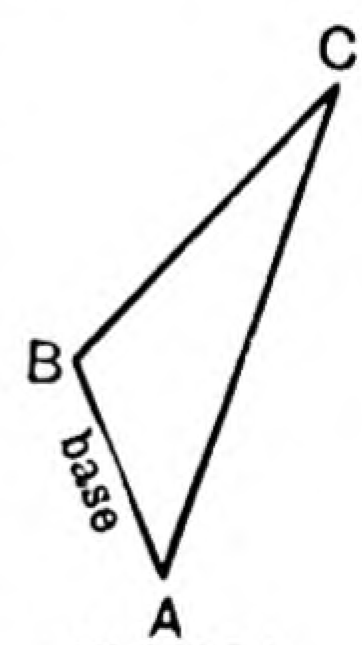


Fig. 114.

$$\mathbf{AC} = \mathbf{AB}\,\frac{\sin B}{\sin C} = \mathbf{AB} \sin B \operatorname{cosec}(A+B).$$

This problem suggests the following practical questions:—

(i) How to measure the distance **AB** ?

(ii) How to observe the angles **CAB, CBA** ?

219. **The base line.**—The measured length **AB** of the last article is called the *base line,* and generally, in surveying, any length which is actually measured with a measuring chain is called a **base line.**

The first operation in any trigonometrical survey consists in measuring a base line somewhere. This is an operation of considerable difficulty. If a measuring chain be used, the chain must be placed exactly in the straight line joining the extreme stations; otherwise the measured distance will be too great. If there be undulating ground between the stations, this condition will be impracticable, and a correction must be made. Moreover, in accurate measurements, a correction must be made for the effect of temperature in causing expansion or contraction of the links of the chain.

On account of these difficulties, only *one* base line is usually measured in trigonometrical surveying, the rest of the work being done entirely by measuring *angles*. (Sometimes *two* base lines are measured with a view of checking any errors of measurement.)

220. **Measurement of angles.**—The angle **BAC** subtended by the line joining two objects **B**, **C** at any place of observation **A** could *theoretically* be observed by a person at **A** first pointing a telescope towards **B** and then pointing it towards **C**, and measuring the angle through which the telescope was turned between the two positions. This is, roughly speaking, the principle of the **theodolite**, an instrument largely used in surveying, for the purpose of measuring angles in a horizontal plane. Again, to measure the angle of elevation or depression of an object as seen from a given station, it is only necessary to point a telescope at the given object and measure the inclination of the telescope to the horizon, the direction of the horizon being found by means of a spirit-level, and this also is another use to which the theodolite can be applied.

Where angles have to be found which are neither in a horizontal nor in a vertical plane, they can be calculated indirectly from observations with a theodolite, or observed directly with a *sextant*.

The method of making observations with a theodolite or sextant belongs to Practical Surveying and not to Trigonometry; for our present purpose we may assume that angles *can* be measured by these instruments, as our object is to show how heights and distances can be calculated from such observations.

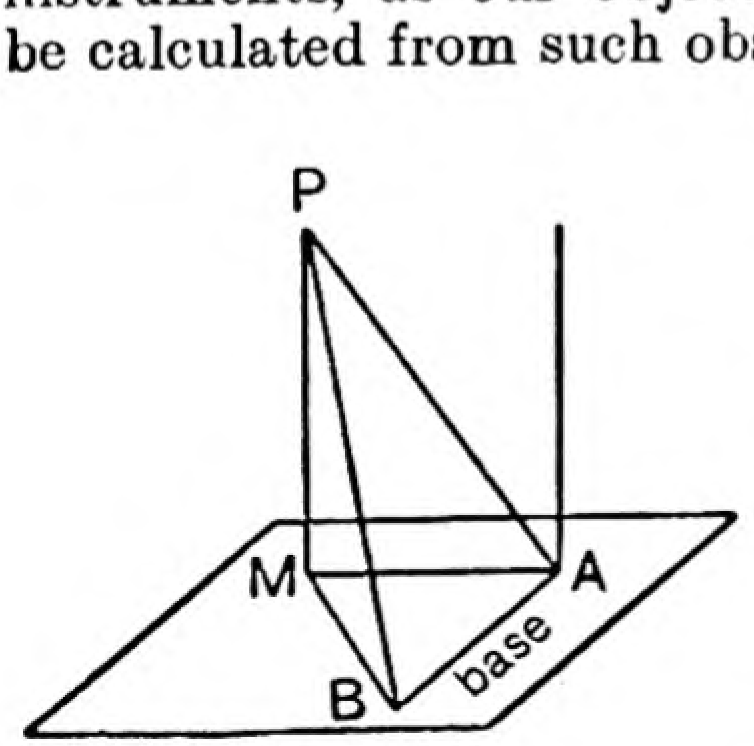


Fig. 115.

221. **To find the height of a mountain or tower.**

Take any station **A** at the foot of the mountain, and in any convenient direction measure a base line **AB**. Let **P** be the summit of the mountain. Observe $\angle$s **PAB, PBA**. Also observe $\angle$**MAP**, the angle of elevation of **P** as seen from **A**.

Then, knowing the base line **AB** and the base angles of the

triangle **PAB**, the distance **AP** may be calculated as in § 218, and **MP**, the vertical height of the mountain, is then given by

$$\mathbf{MP} = \mathbf{AP} \sin \mathbf{MAP}.$$

If $\angle \mathbf{PAB} = \alpha$, $\angle \mathbf{PBA} = \beta$, and if θ denote the angle of elevation **MAP**, it readily follows from this construction that

$$\text{required height } \mathbf{MP} = \mathbf{AB} \sin \beta \sin \theta \operatorname{cosec} (\alpha + \beta).$$

But the construction and not the result should be remembered.

NOTE 1.—In practical observations with a theodolite, supposing **AB** horizontal, the angles observed would not be **PAB** and **PBA**, but **MAB** and **MBA** in a horizontal plane, **M** being vertically below **P**. From these observations, **AM** would be found as in § 218, and then we should have

$$\mathbf{MP} \text{ (the required height)} = \mathbf{AM} \tan \mathbf{MAP}.$$

NOTE 2.—The corresponding problem for the particular case in which the base **AB** is horizontal and in the same vertical plane with **MP** has been given in § 72.

Ex. 1. From a point **A**, the angle of elevation of **P**, the summit of a hill, is observed to be 10° 32′. A base line **AB** is then measured in a convenient direction, 150 yd. 1·76 ft. long, and the angles **PAB**, **PBA** are found to be 46° 15′ and 125° 18′. Find the height of the hill above **A**, using five-figure logarithms.

In the triangle **APB** (Fig. 116), we have

$$\angle \mathbf{APB} = 180° - 46° \, 15' - 125° \, 18' = 8° \, 27',$$

and $\therefore \quad \mathbf{AP} = \mathbf{AB} \sin 125° \, 18' \div \sin 8° \, 27'.$

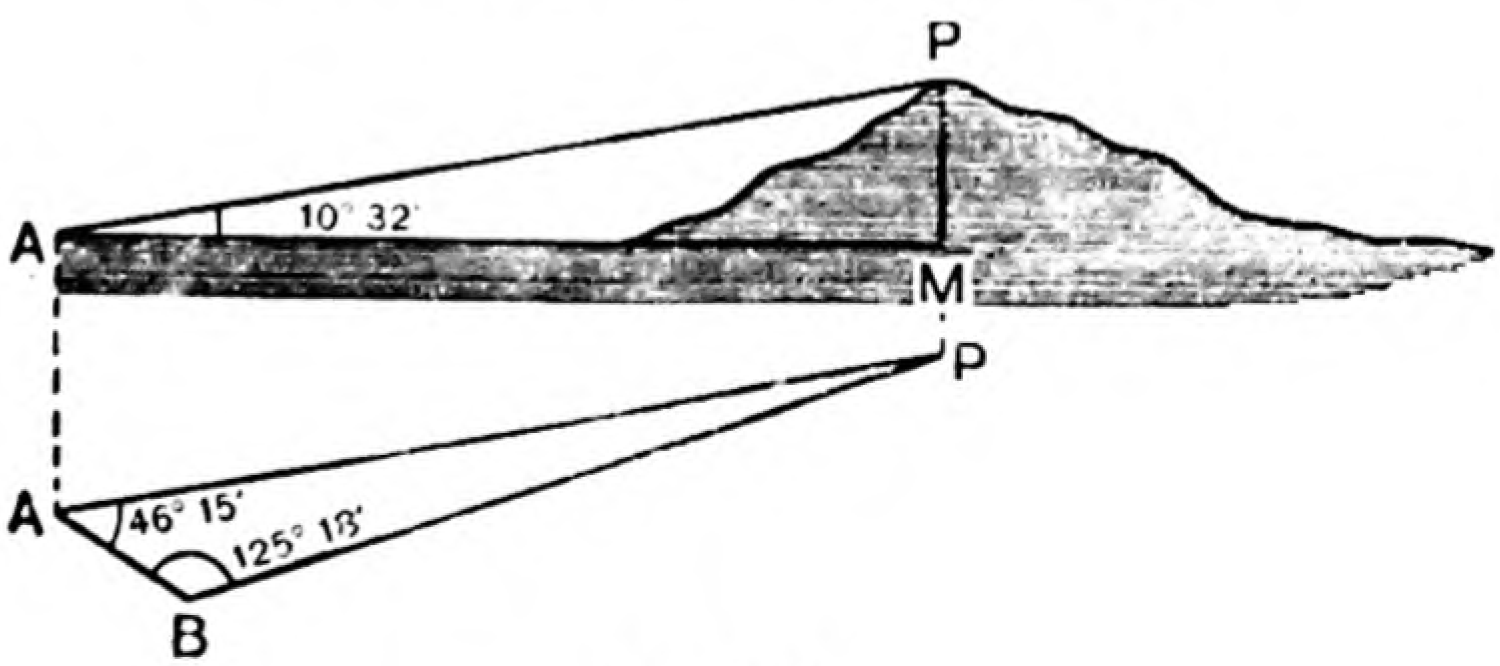


Fig. 116.

Also, if h be the height of the hill in *feet*, the right-angled triangle **MAP** gives

$$h = \mathbf{AP} \sin 10° \, 32'$$
$$= 451{\cdot}76 \sin 10° \, 32' \sin 125° \, 18' \operatorname{cosec} 8° \, 27'.$$

$$\therefore \quad \log h = \log 451{\cdot}76 + L \sin 10° \, 32' + L \sin (180° - 125° \, 18') + L \operatorname{cosec} 8° \, 27' - 30.$$

Now, from the tables,

$$\log 451{\cdot}76 = 2+\log 4{\cdot}5176 = 2{\cdot}65491$$
$$L\sin 10^\circ\, 32' = 9{\cdot}26199$$
$$L\sin 125^\circ\, 18' = L\sin 54^\circ\, 42' = 9{\cdot}91176$$
$$L\,\mathrm{cosec}\, 8^\circ\, 27' = 10{\cdot}83284$$

Adding, we have $\log h = 2{\cdot}66150,$

$$\therefore\quad h = \text{antilog } 2{\cdot}66150 = 458{\cdot}67,$$

or height of hill $= 458{\cdot}67$ ft.

Ex. 2. From the top of a vertical tower which stands on a flat plain, a length a of a flagstaff projects, and is inclined at an angle γ to the horizon. At a point on the ground, in the vertical plane containing the tower and the flagstaff, the elevations of the top of the tower and of the end of the flagstaff are found to be α, β, respectively. Find the height of the tower.

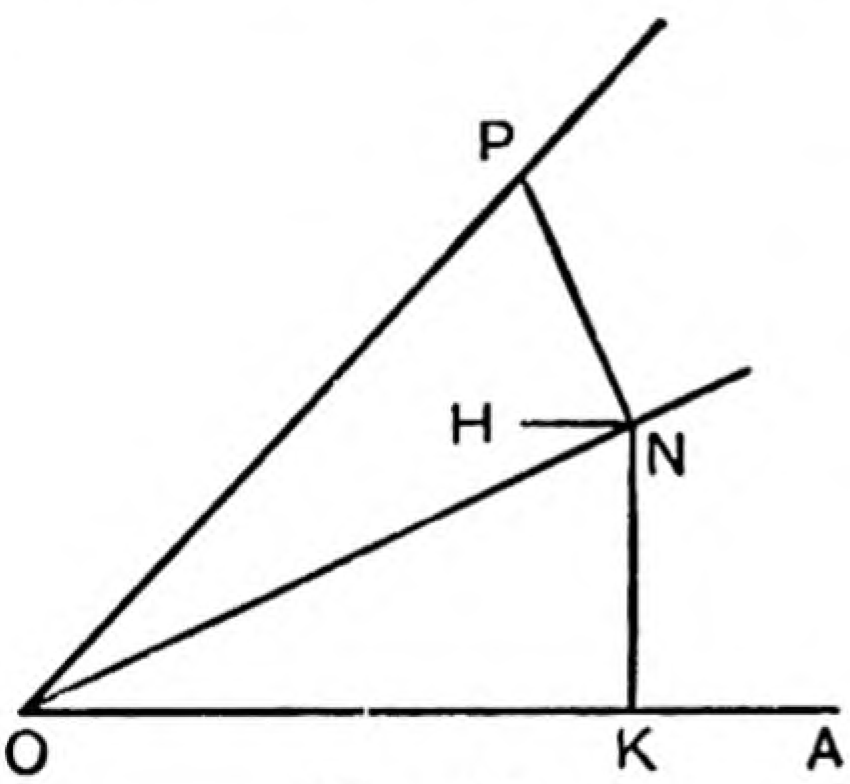


Fig. 117.

Let **KN** be the tower, **NP** the projecting part of the flagstaff, **O** the point of observation.

Then, if the flagstaff makes an acute angle γ with the horizon, and leans *towards* the observer, we have, on drawing **NH** horizontally towards **O**,

$$\angle \mathsf{HNP} = \gamma \text{ and } \angle \mathsf{HNO} = \angle \mathsf{KON} = \alpha,$$

whence $\angle \mathsf{ONP} = \alpha+\gamma;$

also $\angle \mathsf{KOP} = \beta;$

$$\therefore\quad \angle \mathsf{NOP} = \beta-\alpha;$$

$$\therefore\quad \angle \mathsf{OPN} = \pi-(\angle \mathsf{ONP}+\angle \mathsf{NOP})$$
$$= \pi-(\alpha+\gamma+\beta-\alpha) = \pi-(\beta+\gamma).$$

Now $$\mathsf{ON} = \mathsf{NP}\frac{\sin \mathsf{OPN}}{\sin \mathsf{NOP}} = a\frac{\sin\{\pi-(\beta+\gamma)\}}{\sin(\beta-\alpha)}$$
$$= a\sin(\beta+\gamma)\,\mathrm{cosec}\,(\beta-\alpha);$$

$$\therefore\quad \mathsf{KN} = \mathsf{ON}\sin \mathsf{KON} = \mathsf{ON}\sin\alpha$$
$$= a\sin\alpha\sin(\beta+\gamma)\,\mathrm{cosec}\,(\beta-\alpha).$$

If the flagstaff leans *away from* the observer, still making an acute angle γ (on the other side) with the horizontal,

$$\mathsf{KN} = a\sin\alpha\sin(\gamma-\beta)\,\mathrm{cosec}\,(\beta-\alpha).$$

This can be deduced at once from the above, by substituting $\pi-\gamma$ for γ in the result; for in this case the staff makes an angle $(\pi-\gamma)$ with **NH**.

ILLUSTRATIVE EXERCISE.

Verify that the formula obtained in § 72 when put into a form adapted to logarithmic computation leads to a result identical with that of § 221.

222. To find the distance apart of two inaccessible objects.

Let **P, Q** be the objects. We may suppose, *e.g.* that these are on one side of a river, and that an observer on the opposite side requires to find the distance **PQ**, and has no means of crossing the river.

Measure off any convenient base line **AB**. Observe the base angles of $\triangle$**PAB**, and thus calculate **AP**, as in § 218. Observe, similarly, the base angles of $\triangle$**QAB**, and thus calculate **AQ**. Also observe $\angle$**PAQ**.

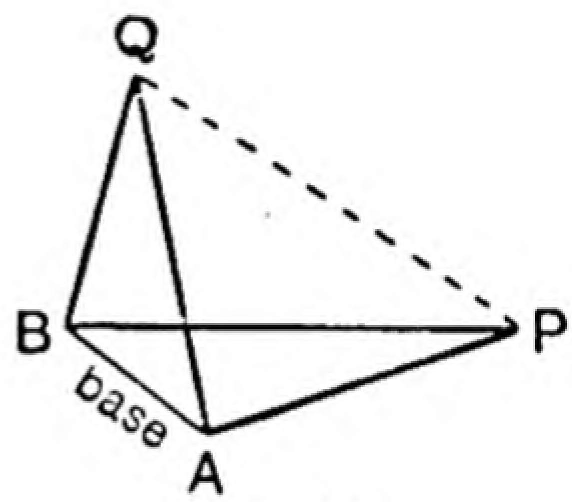


Fig. 118.

Then, in $\triangle$**APQ**, the two sides **AP**, **AQ** and their included angle are known; hence **PQ** can be found. (Case III. of the preceding Chapter.)

[*If the objects are all in one plane*, it will be unnecessary to observe $\angle$**PAQ**, for this will be the difference of the $\angle$s **BAP, BAQ**.]

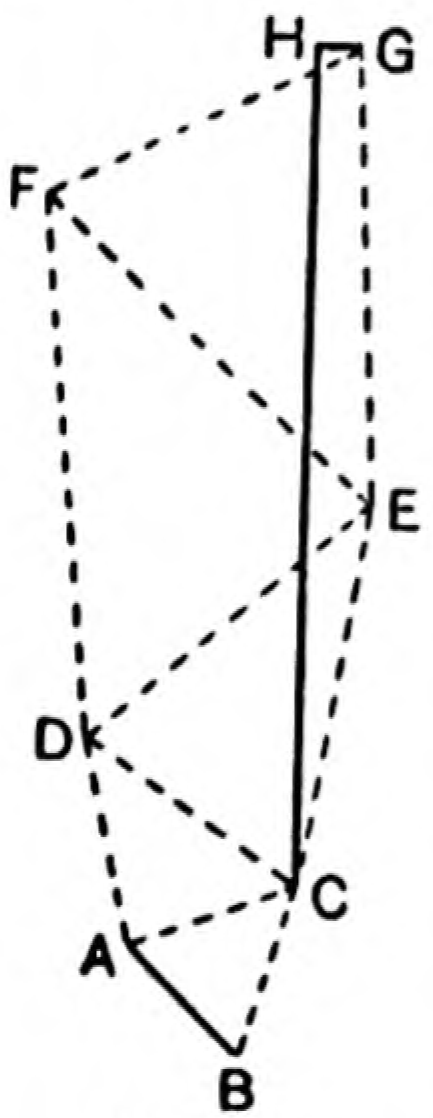


Fig. 119.

223. To perform the survey of any level country from the measurement of a single base line.

Measure a base line **AB** in any convenient place. Take any station **C** visible from both **A** and **B**, and observe the angles of $\triangle$**ABC**. Then, by solving this triangle, the lengths **AC, CB** can be found. We may now take either of these sides as the known base of a new triangle; thus, if **D** be another station, and we observe the angles of $\triangle$**DAC**, the sides **AD, DC** may be found. Similarly, if **E** be any other station, then, by observing the angles of $\triangle$**CDE**, the sides **CE, ED** may now be found. Proceeding in this way, the distance between any two stations **C, H** can be determined by the measurement of angles alone, when once the base line **AB** has been measured.

As an illustration of the accuracy obtainable in the *practical* use of this method, it is recorded that, in an Ordnance Survey of the United Kingdom, *two* base lines were measured—one on Salisbury Plain and the other in Ireland—and when distances were calculated *independently* from the two base lines, the discrepancy (due to errors of observation) was found to amount to not more than a few feet in a length of many miles.

EXAMPLES XX.

1. **ABC** is an equilateral triangle in a horizontal plane; **D** is the point of intersection of the perpendiculars drawn from the angular points to the opposite sides; a flagstaff 20 ft. high is set up at **D**. If a side of the triangle subtends an angle of 60° at the top of the flagstaff, show that the length of the side is $10\sqrt{6}$ ft.

2. A man, travelling due W. along a straight road, observes two objects which have the same bearing, 30° W. of N.; a mile further on one of the objects bears due N. and the other N.E. Find the distances of the objects from the road and from each other.

3. The lengths of the lines joining three points **A, B, C** are observed. At any point **P** in the plane of **ABC**, the angles **APC** and **BPC** are observed. Find the distance of **P** from each of the points **A, B, C**.

4. A vertical pole over 100 ft. high consists of two parts, the lower being $\frac{1}{3}$ of the whole. From a point in the horizontal plane through the foot of the pole and 40 ft. from it, the upper part subtends an angle whose tangent is $\frac{1}{2}$. Find the height of the pole.

5. **A, B,** and **C** are three consecutive milestones on a straight road, from each of which a distant spire is visible. The spire is observed to bear N.E. at **A**, due E. at **B**, and 60° E. of S. at **C**. Find the distance of the spire from **A**, and the shortest distance of the spire from the road.

6. A row-boat is sighted due N. from a steamer, and is pulling E. at the rate of 5 miles an hour. If the steamer's speed be 13 miles an hour, in what direction must she steer in order to come up with the boat at the earliest possible moment?

7. The dip of a stratum is a degrees to the E. Find its apparent dip in a direction b degrees S. of E. Adapt your result to logarithmic calculation.

8. A person walking along a straight road observes that the greatest angle subtended by the line joining two objects is α. From the point where this is the case he walks a yd., and the objects there appear in a straight line, making an angle β with the road. Show that the distance between the objects is $\dfrac{2a \sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$.

9. **A, B, C** are any three points in a horizontal plane. If the distance from **A** to **B** be $100(\sqrt{3}-1)$ yd., and the distance from **A** to **C** be 100 yd., and the angle **BAC** be 60°, find the distance from **B** to **C**.

10. **O** is the centre and **AB** the vertical diameter of a circle, whose plane is vertical and highest point **A**; **C** is a given point in **BC**, a line at right angles to the plane of the circle; **P** is a point on the circumference. Given the radius r, $\mathbf{CB} = a$, and the vertical elevation of **P** at $\mathbf{C} = \theta$, find the inclination of **PO** to **AO**, and the condition that **P** may be in the lower half of the circumference.

11. A statue 30 ft. high, standing on the top of a column, subtends at a point distant 150 ft. in a horizontal line from the base of the column, the same angle as that subtended at the same point by a man 6 ft. high standing at the base of the column. Find the height of the column.

12. A tower subtends an angle θ at a point on the same level with the foot of the tower; and, at a second point h ft. above the former, the angle of depression of the base of the tower is ϕ. Find the height of the tower.

13. The sides of a valley are two parallel hills each of which slopes upwards at an angle of 30°. A man walks 300 yd. directly up one of the hills from the valley, and then observes that the angle of elevation of the top of the other hill above the horizon is 15°. Show that the height of the latter hill is, approximately, 409·8 yd.

14. **A** is a station exactly 10 miles W. of **B**. The bearing of a particular rock from **A** is 74° 19′ E. of N., and its bearing from **B** is 26° 51′ W. of N. How far is it N. of the line joining **A** and **B**?

15. At a point **A** the elevation of a tower is 31° 20′; 1,000 ft. nearer the elevation is 54° 41′: find its height and the distance from **A**.

16. At a distance of 1,300 ft. from the base of a lighthouse, a door which is exactly one-third of its height from the ground has an elevation of 10° 30′. Calculate the height of the lighthouse and the elevation of its top.

17. **AB** is a line 1,000 yd. long; **B** is due N. of **A**. At **B** a distant point **P** bears 70° E. of N.; at **A** it bears 41° 22′ E. of N. Find the distance from **A** to **P**.

18. **AB** is a line 2,000 ft. long; **B** is due E. of **A**. At **B** a distant point **P** bears 46° W. of N.; at **A** it bears 8° 45′ E. of N. Find the distance from **A** to **P**.

19. An observer sees on the opposite side of a stream a tree which subtends an angle of 35° 16′. On walking back 23 ft. he finds that it subtends an angle of 23° 43′. What is the breadth of the stream?

20. **AB** is a line 250 ft. long, in the same horizontal plane as the foot **D** of a tower **CD**; the angles **DAB** and **DBA** are respectively 61° 23′ and 47° 14′; the angle of elevation **CAD** is 34° 50′. Find the height of the tower.

21. If the two sides which include the right angle of a right-angled triangle are of lengths 154 and 231, respectively, find the remaining angles.

22. **A, B, C** are three points in a horizontal plane; the angle **BAC** is a right angle, and the length of **AC** is 1,000 ft.; **P** and **Q** are points vertically over **A** and **B**, and the line joining **P** and **Q** is horizontal; the angle of vertical elevation of **P** at **C** is $52^\circ 40'$, and that of **Q** at **C** is $34^\circ 43'$. Find the distance **PQ**.

23. **A** and **B** are two stations 531 yd. apart; **P** and **Q** are two objects in the same horizontal plane as **A** and **B**. The following angles are found by observation:—

$$ABQ = 127^\circ 35',\ BAQ = 36^\circ 43',\ QAP = 73^\circ 21',\ ABP = 43^\circ 26'.$$

Prove that

$$AQ = 1555{\cdot}1 \text{ yd.},$$

$$AP = 818{\cdot}2 \text{ yd.},\quad PQ = 1535{\cdot}7 \text{ yd.}$$

24. At a point on a horizontal plain, the elevation above the horizontal of the summit of a mountain is observed to be $22^\circ 15'$, and at another point on the plain, a mile further away in a direct line, its elevation is observed to be $10^\circ 12'$. Calculate the height of the mountain in feet.

25. From the extremities of a base line **AB**, whose length is 1,125 ft., the bearings of the foot **C** of a tower are observed, and it is found that

$$\angle CAB = 56^\circ 23',\quad \angle CBA = 47^\circ 15',$$

$$\text{elevation of tower from } A = 9^\circ 25'.$$

Calculate the height of the tower.

26. An observer, wishing to know the distance from a point **C** on one side of a stream to an object **A** on the other side, measures a base line **CB**, 250 yd. long. He then observes the angle **ABC** to be $14^\circ 15'$, and the angle **ACB** to be $59^\circ 31'$. What is the distance?

27. A wall 20 ft. high bears $59^\circ 5'$ E. of S.; find the width of its shadow on a horizontal plane at the instant that the sun is due S. at an altitude of 30°.

28. The altitude of a mountain, observed at the end **A** of a base line **AB** of $2992{\cdot}5$ ft., was $19^\circ 42'$, and the horizontal angles at **A** and **B** were $127^\circ 54'$ and $33^\circ 9'$. Find the height of the mountain.

CHAPTER XXI.

THE CIRCLES OF A TRIANGLE.

224. Definition.—The circle which passes through the three vertices of a triangle is called the **circumscribing circle** of the triangle.

This name is sometimes contracted into *circumcircle*, and the centre and radius of the circle are then spoken of as the *circumcentre* and *circumradius* of the triangle.

The geometrical construction for the circle circumscribing a triangle is given in Euclid IV. 5, from which we learn that the circumcentre is the point of intersection of lines drawn perpendicular to the sides of the triangle through their middle points. We shall now prove that, if R be the radius of the circumcircle,

$$R = \frac{a}{2 \sin A} = \frac{abc}{4S},$$

where S is the area of the triangle.

225. To find the radius of the circle circumscribing a triangle.

Let **O** be the centre of the circumscribing circle.

Join **CO**, and produce it to cut the circle in **D**. Join **DB**.

Then $\angle$**DBC** (being the angle in a semicircle) $= 90°$.

If A be acute,

$$\angle \mathbf{BDC} = \angle \mathbf{BAC} \text{ in same segment} = A.$$

Now, $$\sin \mathbf{BDC} = \frac{\mathbf{BC}}{\mathbf{DC}};$$

but $\mathbf{DC}$ = diameter of circle $= 2R$ (where R = radius);

$$\therefore \quad \sin A = \frac{a}{2R};$$

$$\therefore \quad \boldsymbol{R} = \frac{\boldsymbol{a}}{\mathbf{2 \sin} \boldsymbol{A}} \quad \text{..................... }\mathbf{(115)}$$

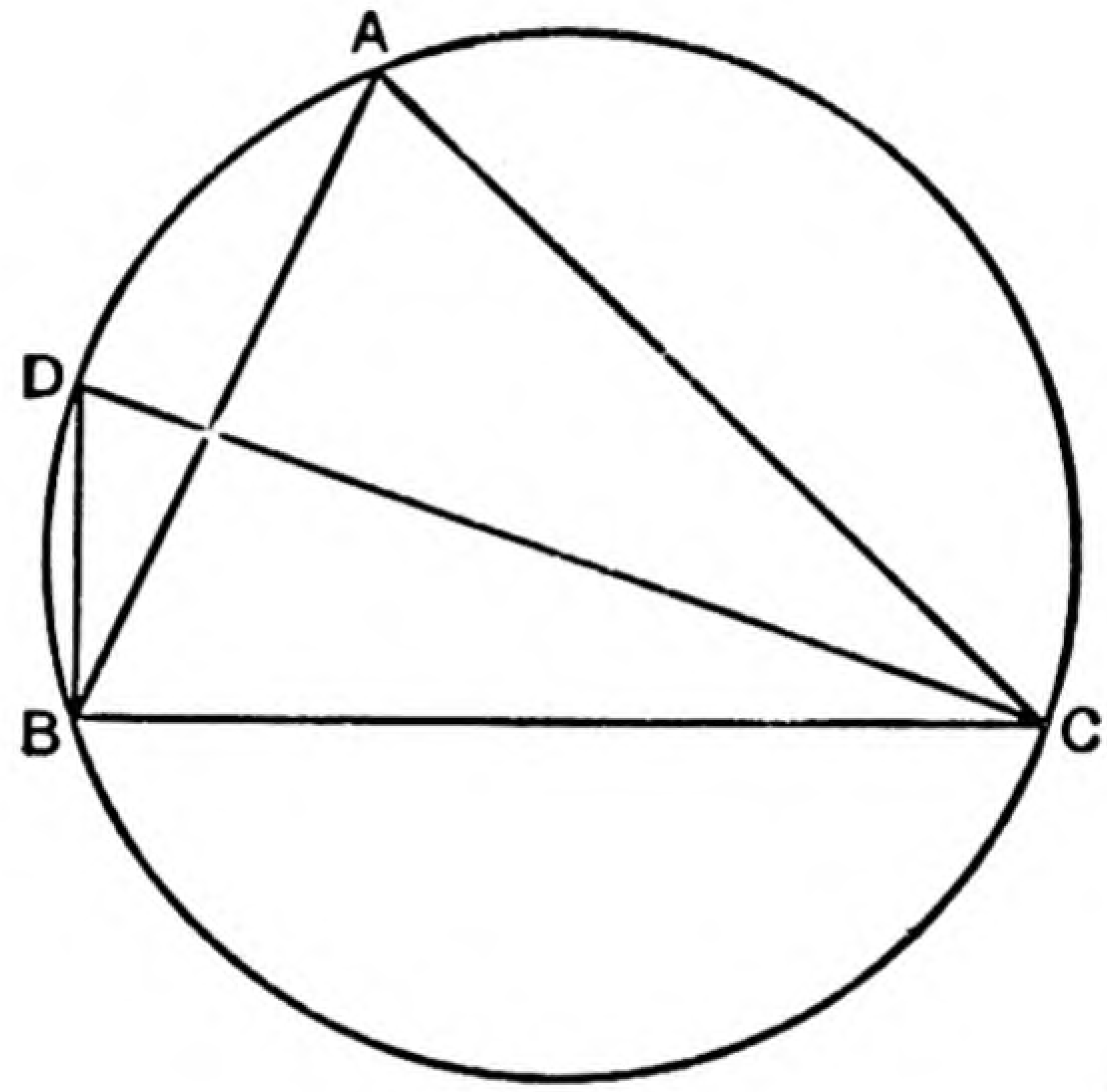


Fig. 120.

If A be obtuse, we shall find that **O** lies outside the triangle, and **A, D** are on opposite sides of **BC**.

In this case, $\angle \mathbf{BDC} = 180° - A$, and we obtain

$$R = \frac{a}{2 \sin (180° - A)} = \frac{a}{2 \sin A},$$

as before.

Again, $$\sin A = \frac{2}{bc} S,$$

where S is the area of the triangle (§ 208);

$$\therefore \boldsymbol{R} = \frac{\boldsymbol{abc}}{\mathbf{4}\boldsymbol{S}} \quad \text{...................... }\mathbf{(116)}$$

COR.—From the Principle of Symmetry, we deduce that

$$R = \frac{b}{2 \sin B} \quad \text{and} \quad R = \frac{c}{2 \sin C}.$$

We thus have an alternative proof of the sine rule, which we may now write in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{ ....(116a)}$$

Ex. 1. To prove that $R = \frac{1}{4}s \sec \frac{1}{2}A \sec \frac{1}{2}B \sec \frac{1}{2}C.$

Since $a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C,$

$$s = \tfrac{1}{2}(a+b+c) = R(\sin A + \sin B + \sin C)$$
$$= 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$

(§ 144, Ex. 1, since $A+B+C = 180°$);

$$\therefore \quad R = \tfrac{1}{4}s \sec \tfrac{1}{2}A \sec \tfrac{1}{2}B \sec \tfrac{1}{2}C.$$

Ex. 2. To express the area of a triangle in terms of R and the angles.

$$S = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}(2R \sin B)(2R \sin C) \sin A;$$
$$\therefore \quad S = 2R^2 \sin A \sin B \sin C.$$

226. **Inscribed and Escribed Circles.**—DEFINITIONS: The circle which touches all three sides of a triangle and lies within the triangle is called the **inscribed circle** of the triangle.

The contractions *incircle, incentre,* and *inradius* are sometimes used to denote this circle and its centre and radius.

The geometrical construction for this circle is given in Euclid IV. 4, from which we learn that the incentre is the point of intersection of the three bisectors of the angles of the triangle.

Besides this circle, three circles can be constructed, each of which touches one of the sides of the triangle and the other two sides *produced*. These circles are called the **escribed circles** of the triangle.

The contractions *excircle, excentre,* and *exradius* are used in connection with these circles.

The centre I_1 of the escribed circle opposite the angle **A** (*i.e.* the circle which touches the side a and the other two sides produced) is the common point of intersection of the bisector of the interior angle of the triangle at **A** and the exterior angles at **B** and **C.** That is, I_1A bisects $\angle BAC$ and I_1B, I_1C bisect the angles between **BC** and the produced directions of **AB, AC.**

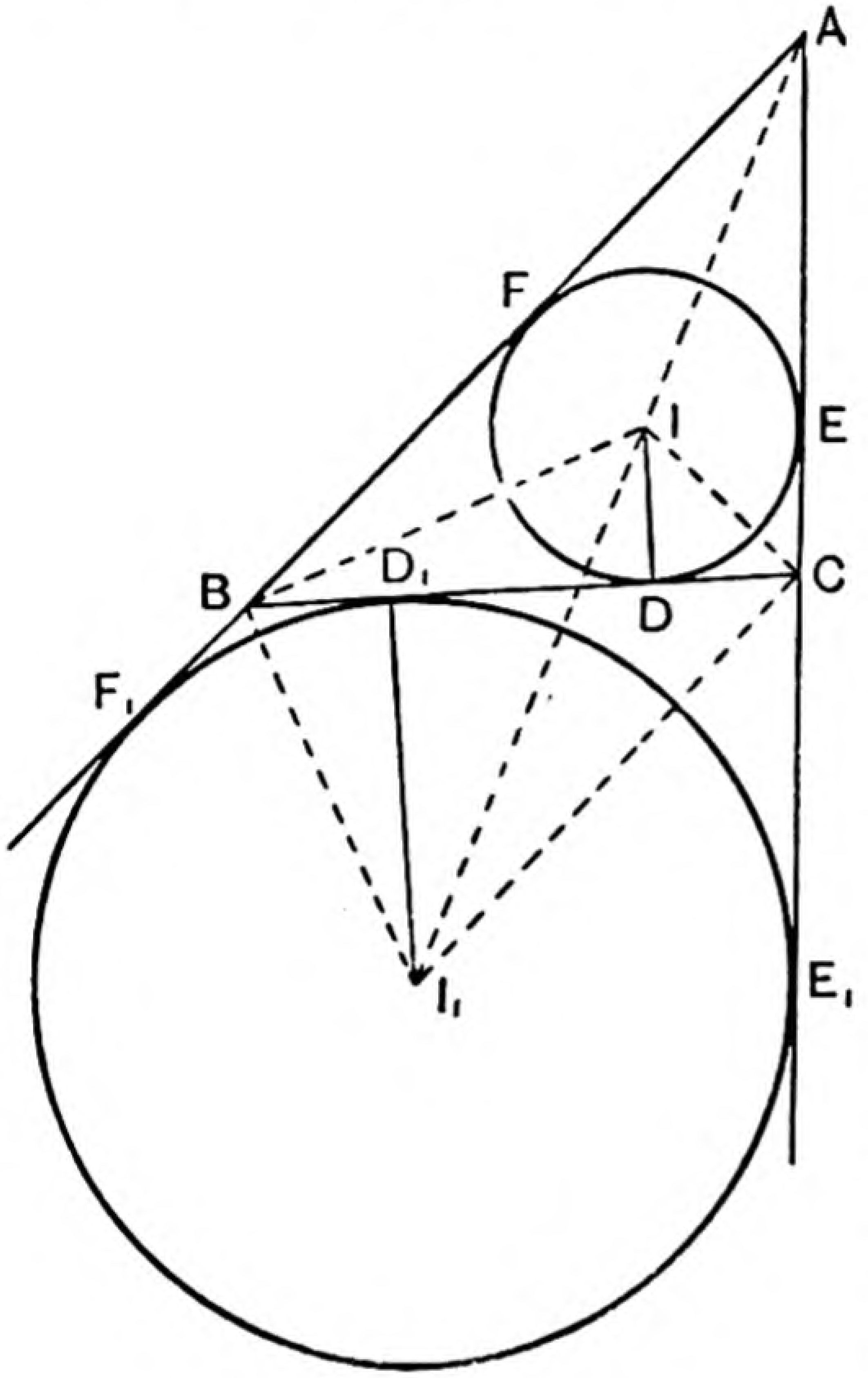


Fig. 121.

We shall now prove that, if r be the radius of the inscribed circle, r_1, r_2, r_3 the radii of the three escribed circles, and S, s have their usual meanings,

$$r=\frac{S}{s},\quad r_1=\frac{S}{s-a},\quad r_2=\frac{S}{s-b},\quad r_3=\frac{S}{s-c}.$$

NOTE.—The letters R, r, r_1, r_2, r_3 are almost universally used to denote the radii of the circum-, in-, and three escribed circles of a triangle.

227. To find the radius of the circle inscribed in a triangle.

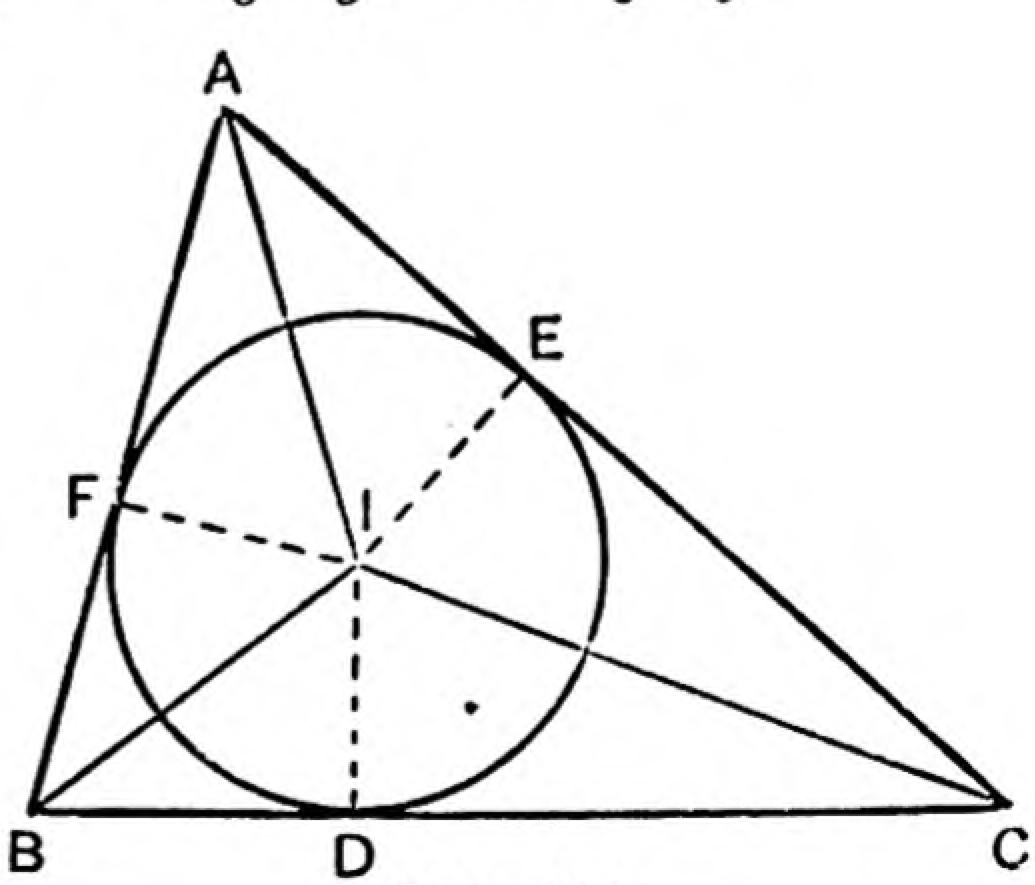


Fig. 122.

Let **I** be the centre of the circle. Join **IA, IB, IC,** and draw **ID** perpendicular to **BC.** Then **D** is the point of contact of circle, and **ID** is its radius $= r$.

Hence $\qquad$ area $\mathsf{IBC} = \frac{1}{2}\mathsf{BC}.\mathsf{ID}$

$$= \tfrac{1}{2}ar;$$

similarly, $\qquad$ area $\mathsf{ICA} = \frac{1}{2}br$,

$\qquad$ area $\mathsf{IBA} = \frac{1}{2}cr$.

Adding, we have

$$\text{area } \mathsf{ABC} = \tfrac{1}{2}(a+b+c)\,r;$$

$$\therefore \quad S = sr,$$

and
$$r = \frac{S}{s} \quad \text{......................(117}$$

228. To find the radii of the circles escribed to a triangle.

Let I_1 be the centre of the circle escribed opposite the angle A. Join $\mathsf{I}_1\mathsf{A}$, $\mathsf{I}_1\mathsf{B}$, $\mathsf{I}_1\mathsf{C}$, and draw $\mathsf{I}_1\mathsf{D}_1$ perpendicular on BC. Then $\mathsf{I}_1\mathsf{D}_1$ = radius of escribed circle $= r_1$, and we have

area $\mathsf{I}_1\mathsf{BC} = \frac{1}{2}r_1 a$,

area $\mathsf{I}_1\mathsf{CA} = \frac{1}{2}r_1 b$,

area $\mathsf{I}_1\mathsf{AB} = \frac{1}{2}r_1 c$;

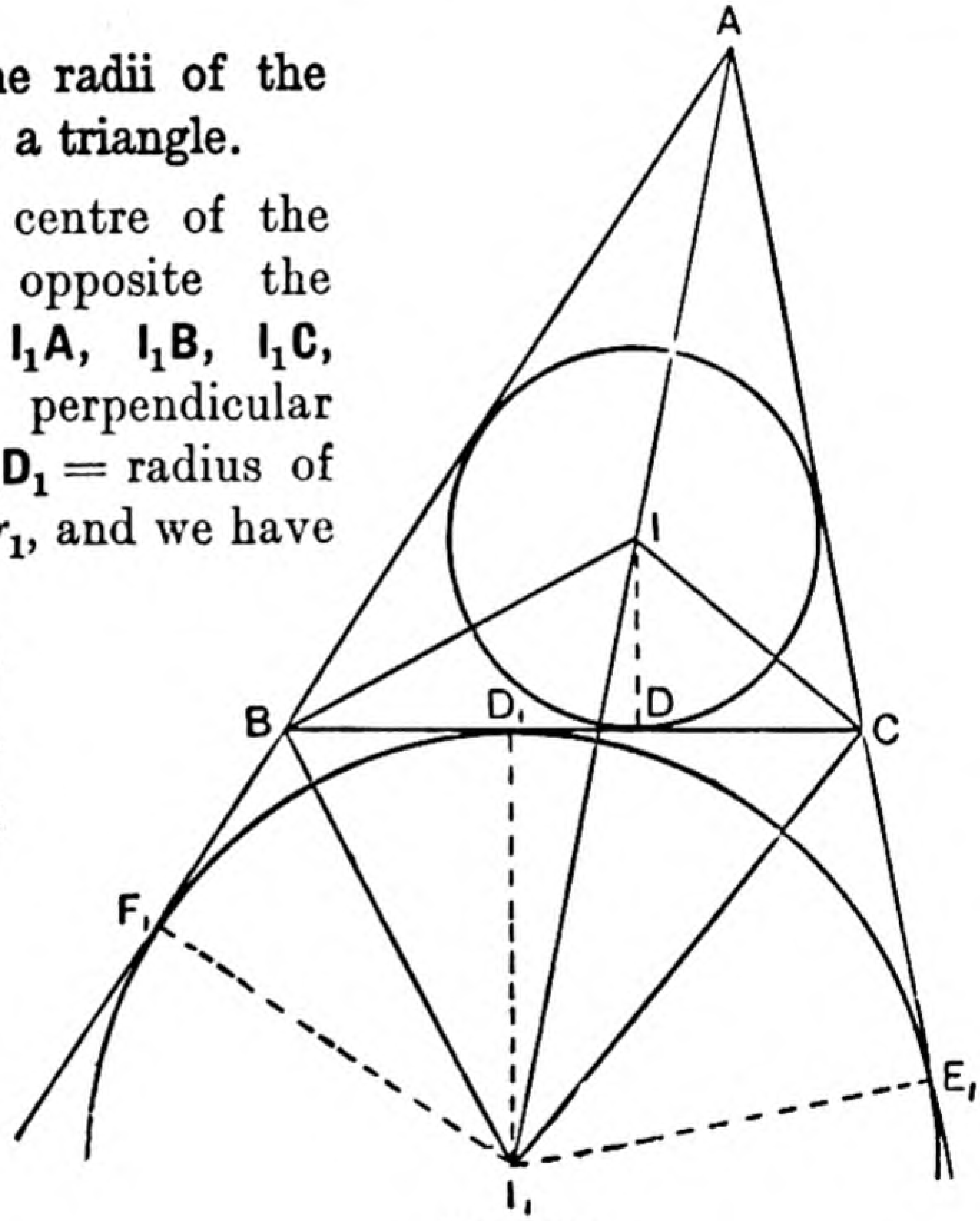


Fig. 123.

$$\therefore \quad \text{area } \mathsf{ABC} = \mathsf{I}_1\mathsf{AB} + \mathsf{I}_1\mathsf{AC} - \mathsf{I}_1\mathsf{BC}$$

$$= \tfrac{1}{2}r_1(b+c-a);$$

$$\therefore \quad S = r_1(s-a),$$

and
$$r_1 = \frac{S}{s-a}.$$

Writing down by symmetry the radii of the two other escribed circles, we have the three formulae

$$r_1 = \frac{S}{s-a}, \quad r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c} \quad \text{............(118)}$$

Ex. 1. $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}.$$

For $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a}{S}+\frac{s-b}{S}+\frac{s-c}{S}=\frac{3s-(a+b+c)}{S}$$
$$=\frac{3s-2s}{S}=\frac{s}{S}=\frac{1}{r}.$$

Ex. 2. $$rr_1r_2r_3 = S^2.$$

For $$rr_1r_2r_3 = \frac{S}{s}\cdot\frac{S}{s-a}\cdot\frac{S}{s-b}\cdot\frac{S}{s-c}=\frac{S^4}{S^2}=S^2.$$

This identity shows that the area of a triangle when expressed in terms of the radii of its inscribed and escribed circles is equal to $\sqrt{(rr_1r_2r_3)}$.

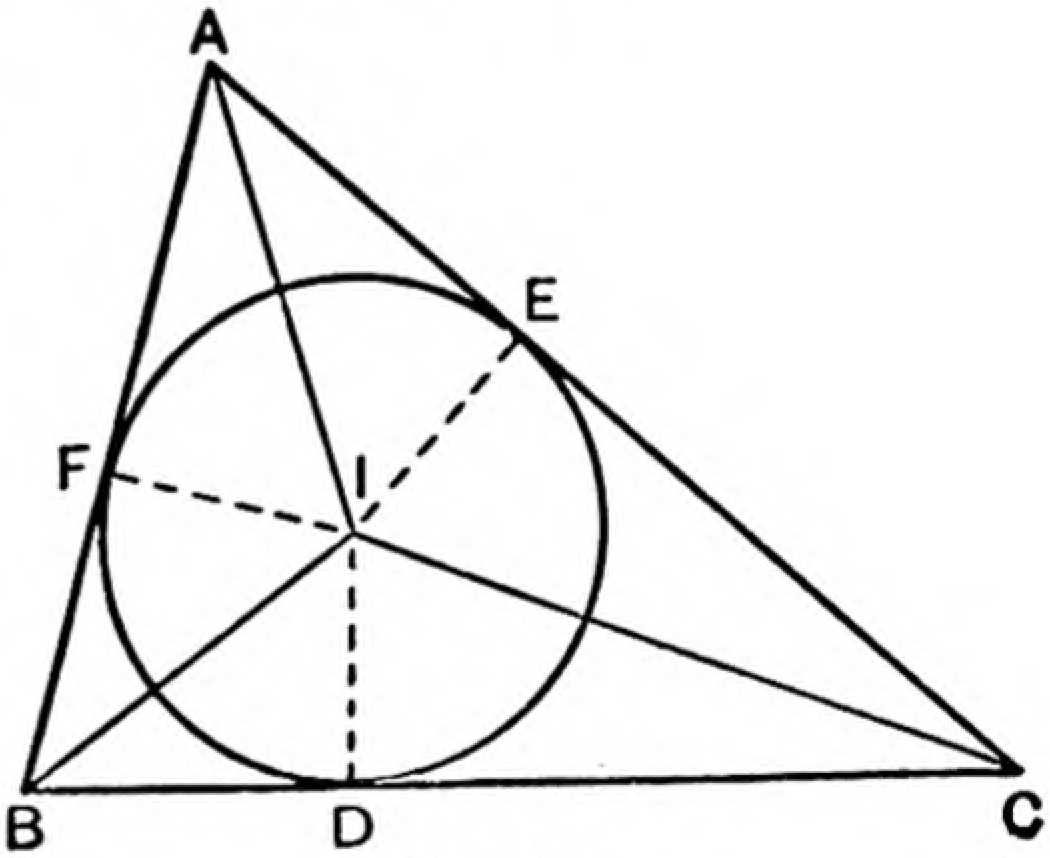


Fig. 124.

229. To express the radius of the inscribed circle in terms of a side and the angles.

In Fig. 124 we have

$$\angle \mathbf{IBC} = \tfrac{1}{2}B, \quad \angle \mathbf{ICB} = \tfrac{1}{2}C,$$

$$\mathbf{DI}\cot \mathbf{IBC} + \mathbf{DI}\cot \mathbf{ICB} = \mathbf{BD}+\mathbf{DC} = \mathbf{BC};$$

$$\therefore \quad r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C) = a;$$

$$\therefore \quad r = \frac{a}{\cot\frac{1}{2}B+\cot\frac{1}{2}C} = \frac{a\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}(B+C)}$$

$$\therefore r = a\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\sec\tfrac{1}{2}A \text{.................(119)}$$

This result may also be obtained thus—

$$r = \mathbf{BI}\sin\tfrac{1}{2}B \quad \text{and} \quad \mathbf{BI} = \mathbf{BC}\sin\mathbf{BCI}/\sin\mathbf{BIC}.$$

But $$\angle\mathbf{BIC} = 180^\circ - \tfrac{1}{2}B - \tfrac{1}{2}C = 90^\circ + \tfrac{1}{2}A;$$

$$\therefore \quad r = a\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C/\sin(90^\circ+\tfrac{1}{2}A) = a\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\sec\tfrac{1}{2}A.$$

230. Relation between the radii of the inscribed and circumscribed circles.

Since $$a = 2R \sin A = 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A,$$
the result of the last article gives the convenient formula
$$r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \quad \text{................(120)}$$

231. To express the radii of the escribed circles in terms of a side and the angles.

We shall now deduce expressions for the radii of the escribed circles analogous to those established in §§ 229, 230 for the radius of the inscribed circle.

In Fig. 125, $\angle \mathsf{I_1BC} = \tfrac{1}{2}\mathsf{F_1BC}$
$$= \tfrac{1}{2}(180° - B)$$
$$= 90° - \tfrac{1}{2}B,$$
$$\angle \mathsf{I_1CB} = 90° - \tfrac{1}{2}C,$$
and $$\angle \mathsf{BI_1C} = 180° - \mathsf{I_1BC} - \mathsf{I_1CB}$$
$$= \tfrac{1}{2}(B + C) = 90° - \tfrac{1}{2}A;$$

Then $$\mathsf{BC} = \mathsf{I_1D_1} \cot \mathsf{I_1BC} + \mathsf{I_1D_1} \cot \mathsf{I_1CB}$$
or $$a = r_1 \tan \tfrac{1}{2}B + r_1 \tan \tfrac{1}{2}C;$$
$$\therefore \quad r_1 = \frac{a}{\tan \frac{1}{2}B + \tan \frac{1}{2}C}$$
$$= \frac{a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\sin \frac{1}{2}(B+C)},$$
or $$r_1 = \frac{a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A} \quad \text{.........(121)}$$

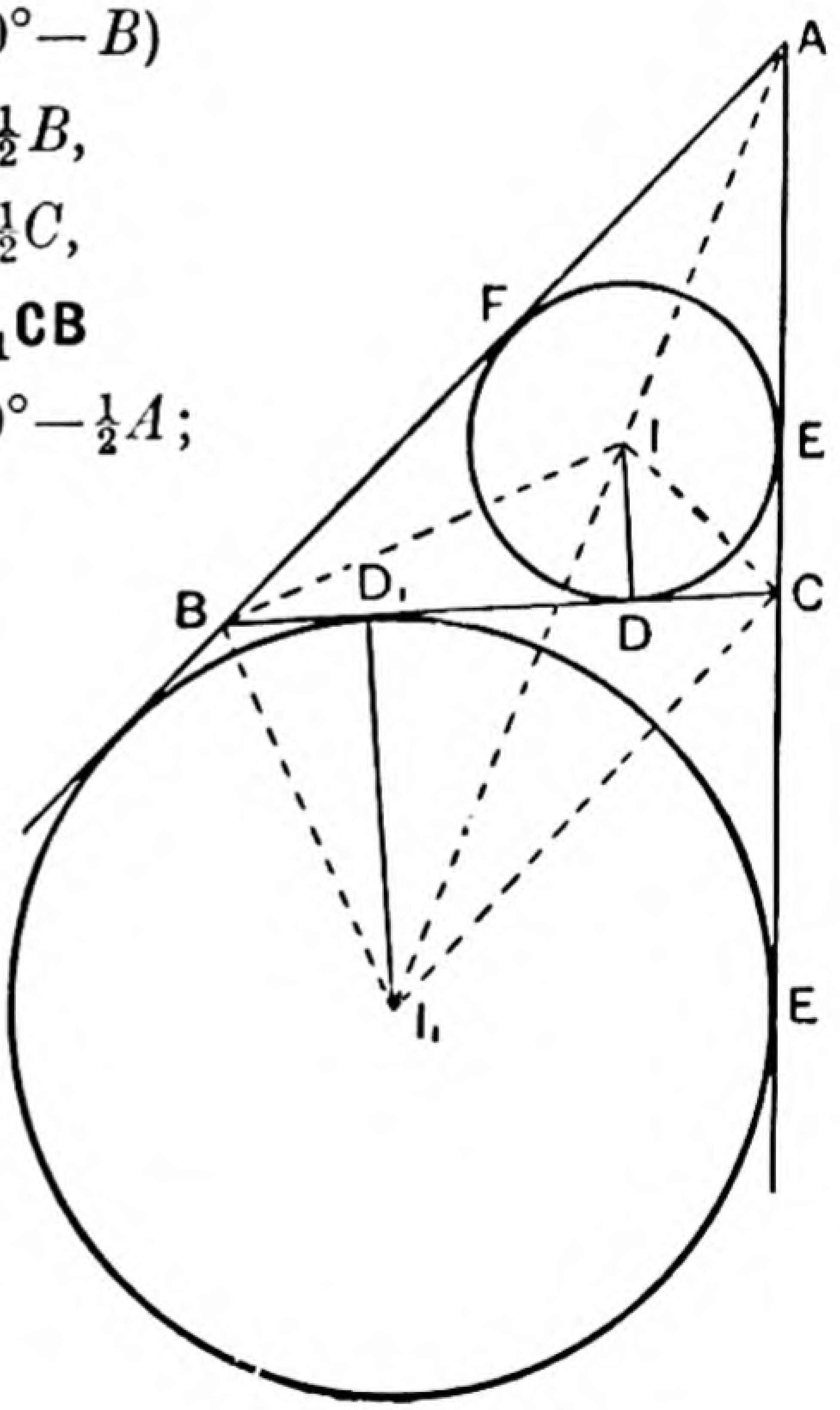


Fig. 125.

The similar expressions for r_2, r_3 should be written down by the student from the principle of symmetry.

Similarly, from the triangles I_1CA, I_1BA, we have

$$r_1 = \frac{b \sin \frac{1}{2}A \cos \frac{1}{2}C}{\sin \frac{1}{2}B} = \frac{c \sin \frac{1}{2}A \cos \frac{1}{2}B}{\sin \frac{1}{2}C}$$

(and similar expressions for r_2, r_3).

But the two last relations could be deduced at once from the first by applying the sine rule.

232. Since $a = 2R \sin A$, we find

$$\left.\begin{aligned} r_1 &= 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C. \\ r_2 &= 4R \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C \cos \tfrac{1}{2}A, \\ r_3 &= 4R \sin \tfrac{1}{2}C \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B. \end{aligned}\right\} \qquad \text{Similarly,} \qquad \ldots\ldots\ldots\ldots(122)$$

These may be conveniently remembered in conjunction with formula (120), $r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$.

233. To find the segments into which the sides are divided by the points of contact of the inscribed circle.

Lettering the points of contact as in Fig. 125, we shall now prove that

$$\left.\begin{aligned} AE = AF &= s-a \\ BF = BD &= s-b \\ CD = CE &= s-c \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(123)$$

For, since the two tangents from a point to a circle are equal,

$$AE = AF, \quad BF = BD, \quad CE = CD;$$

$$\therefore \quad a = BD + DC = BF + CE,$$

and $\quad a + 2AF = BF + CE + AF + AE = AB + AC = b + c;$

$$\therefore \quad 2AF = b + c - a = 2(s-a),$$

and $\quad AF = s - a.$

Similarly, the other relations can be proved. Noticing that **AE, AF** are tangents to the inscribed circle, we may remember these results in the following symmetrical form:—

The lengths of the tangents to the inscribed circle from **A, B, C** *are* $s-a$, $s-b$, $s-c$, *respectively.*

234. To find the segments of the sides made by the escribed circles.

If D_1, E_1, F_1 be the points of contact of the circle opposite the angle A, we have

$$AF_1 = AB + BF_1 = AB + BD_1,$$
$$AE_1 = AC + CE_1 = AC + CD_1;$$
$$\therefore \quad AF_1 + AE_1 = AB + AC + BC = b + c + a = 2s.$$

But $AF_1 = AE_1$. Hence each must be equal to s, that is,

$$AF_1 = AE_1 = s \qquad \text{......................(124)}$$

Hence *the length of the tangent to any escribed circle from the vertex opposite it is s.*

Again,
$$\left.\begin{aligned} BD_1 &= BF_1 = AF_1 - AB = s - c \\ CD_1 &= CE_1 = AE_1 - AC = s - b \end{aligned}\right\} \qquad \text{...........(125)}$$

Cor.—Comparing these results with those of (123), § 233, we see that BC is divided in D and D_1, so that

$$BD_1 = DC \text{ and } BD = D_1C,$$

[Note.—The result of this corollary combined with the Principle of Symmetry enables us to write down with certainty the segments of any side made by the corresponding escribed circle. Thus, from symmetry, CD_1 is not $s-a$, because, if it were, BD_1 would also be $s-a$ (as would follow from interchanging B and C); also, CD_1 is not $s-c$, because $s-c$ is the tangent CD to the *inscribed* circle, and the points of contact of the inscribed and escribed circles do not generally coincide. This leaves us no doubt in writing down $CD_1 = s-b$.]

Ex. 1. The area of a triangle is the square root of the product of the lengths of the tangents from one of its vertices to the inscribed and three escribed circles.

For the lengths of the four tangents from A will be found to be $s-a$, s, $s-b$, $s-c$, and the area

$$= \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Ex. 2.
$$FF_1 = EE_1 = a.$$
For
$$FF_1 = AF_1 - AF = s - (s-a) = a.$$

235. To prove geometrically that the area of a triangle is

$$\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

If S denote the area, we have

$$S = rs \quad \text{and} \quad S = r_1(s-a); \qquad (\S\S\ 227,\ 228)$$
$$\therefore \quad S^2 = s(s-a)\, rr_1.$$

Now, in Fig. 125, **BI** and **$\mathbf{BI_1}$** are the internal and external bisectors of the angles at **B**, and are therefore at right angles. Hence the triangles **$\mathbf{BD_1I_1}$** and **IDB** are similar;

$$\therefore \quad \frac{\mathsf{D_1I_1}}{\mathsf{BD_1}} = \frac{\mathsf{BD}}{\mathsf{DI}}, \quad \text{or} \quad \frac{r_1}{s-c} = \frac{s-b}{r};$$

$$\therefore \quad rr_1 = (s-b)(s-c) \text{ and } S^2 = s(s-a)(s-b)(s-c).$$

The proof is purely geometrical; for, in proving that

$$S = rs = r_1(s-a),$$

$\mathsf{BD} = s-b$ and $\mathsf{BD_1} = s-c$, no trigonometrical formulae have been used.

Cor.—In the course of this proof we have established the identity $rr_1 = (s-b)(s-c)$. Similarly, it can be proved that $r_2r_3 = s(s-a)$.

Ex. 1. To prove geometrically that $\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)}$.

In Fig. 125 $\qquad \tan\frac{A}{2} = \frac{\mathsf{FI}}{\mathsf{AF}}$ and $\tan\frac{A}{2} = \frac{\mathsf{F_1I_1}}{\mathsf{AF_1}}$;

$$\therefore \quad \tan^2\frac{A}{2} = \frac{\mathsf{FI}\,.\,\mathsf{F_1I_1}}{\mathsf{AF}\,.\,\mathsf{AF_1}}.$$

But $\triangle$s **FIB**, **$\mathbf{BF_1I_1}$** are similar; $\quad \therefore \quad \mathsf{FI}\,.\,\mathsf{F_1I_1} = \mathsf{BF}\,.\,\mathsf{BF_1}$;

$$\therefore \quad \tan^2\frac{A}{2} = \frac{\mathsf{BF}\,.\,\mathsf{BF_1}}{\mathsf{AF}\,.\,\mathsf{AF_1}} = \frac{(s-b)(s-c)}{(s-a)s}.$$

Ex. 2. Given r_1, r_2, r_3 to find A, B, C.

We have
$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \qquad \text{(§ 228, Ex. 1)}$$
which gives r, and then
$$\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)} = \frac{rr_1}{r_2r_3}. \qquad \text{(§ 235, Cor.)}$$

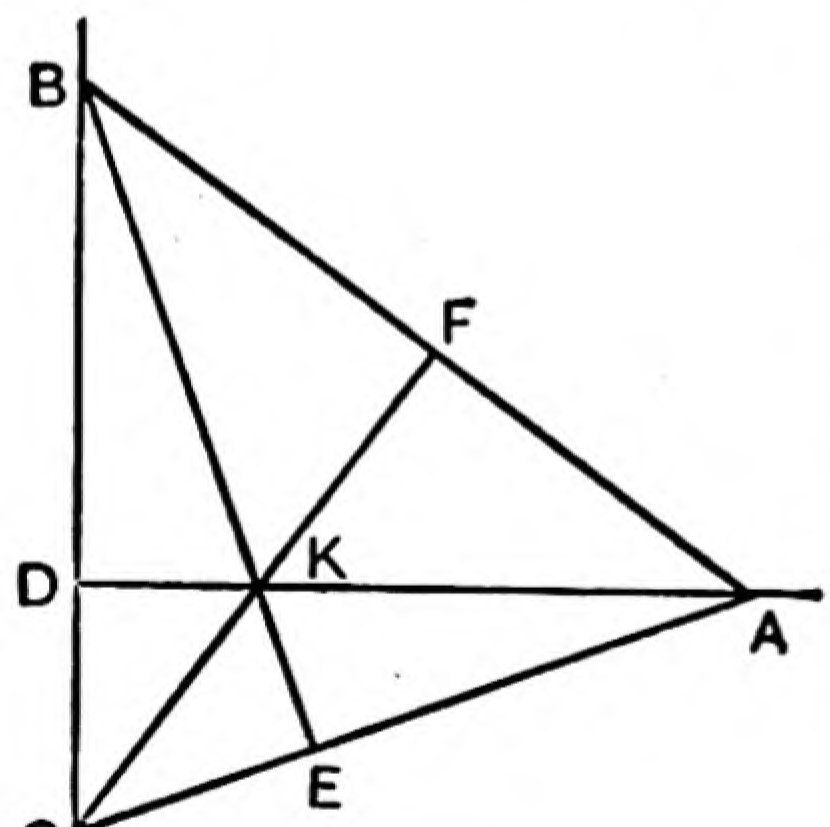


Fig. 126.

236. Properties of the orthocentre of a triangle.—The student is doubtless familiar with the geometrical proof that the three perpendiculars from the vertices of a triangle on the opposite sides pass through one common point (called the *orthocentre* of the triangle).

If **K** be the orthocentre of the $\triangle$ **ABC**, the following properties may be noted:—

$$\mathsf{AK} = \mathsf{AE} \sec \mathsf{EAK}$$
$$= \mathsf{AB} \cos A \operatorname{cosec} C$$
$$= \frac{c}{\sin C}\cos A;$$

$$\therefore \quad \mathsf{AK} = 2R\cos A, \quad \mathsf{BK} = 2R\cos B, \quad \mathsf{CK} = 2R\cos C \ldots\ldots(126)$$

Hence also $AK^2 = 4R^2-a^2$, $BK^2 = 4R^2-b^2$, $CK^2 = 4R^2-c^2$.

Again, $DK = BD \tan DBK = AB \cos B \cot C = \frac{c}{\sin C} \cos B \cos C$;

$$\therefore \quad DK = 2R \cos B \cos C, \quad EK = 2R \cos C \cos A,$$
$$FK = 2R \cos A \cos B \text{ ....} (127)$$

237. To find the distance between the centres of the inscribed and circumscribing circles of a triangle.

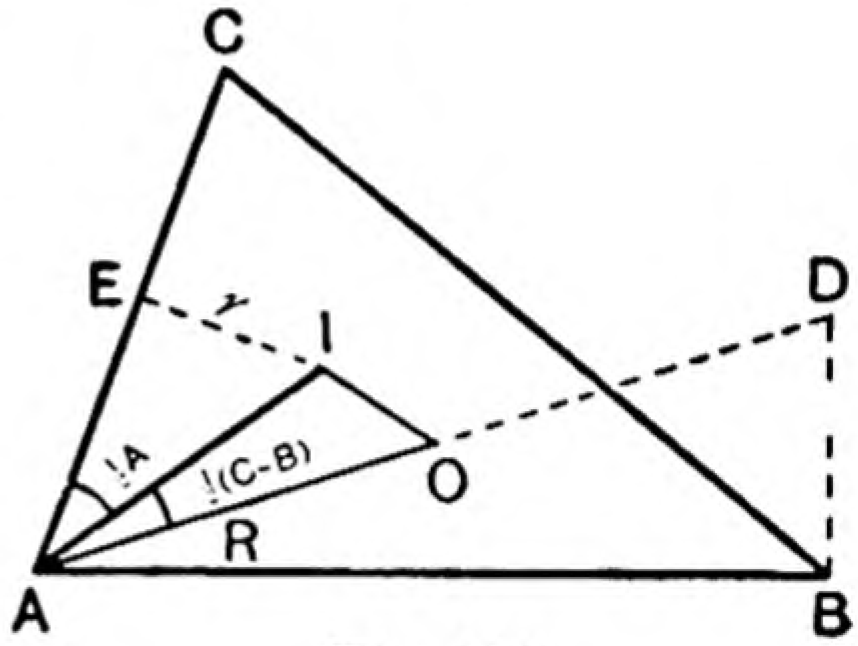


Fig. 127.

If AO meets the circumcircle again in D, we have

$$\angle BAO = 90°-\angle ADB = 90°-C.$$

Similarly,

$$\angle CAO = 90°-B.$$

Again, AI bisects the $\angle BAC$;

$$\therefore \quad \angle OAI = \tfrac{1}{2}\{\angle OAC-\angle OAB\}$$
$$= \tfrac{1}{2}(C-B).$$

Also $AO = R$,

$$AI = r \operatorname{cosec} EAI = r \operatorname{cosec} \tfrac{1}{2}A;$$

$$\therefore \quad OI^2 = AO^2+AI^2-2AO.AI \cos OAI$$
$$= R^2+r^2 \operatorname{cosec}^2 \tfrac{1}{2}A-2Rr \operatorname{cosec} \tfrac{1}{2}A \cos \tfrac{1}{2}(C-B).$$

Now, from § 230, $r \operatorname{cosec} \frac{1}{2}A = 4R \sin \frac{1}{2}B \sin \frac{1}{2}C$;

$$\therefore \quad OI^2 = R^2-r \operatorname{cosec} \tfrac{1}{2}A \;\{2R \cos \tfrac{1}{2}(C-B)-4R \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C\}$$
$$= R^2-2Rr \operatorname{cosec} \tfrac{1}{2}A\{\cos \tfrac{1}{2}(C-B)-\cos \tfrac{1}{2}(C-B) + \cos \tfrac{1}{2}(C+B)\}$$
$$= R^2-2Rr \operatorname{cosec} \tfrac{1}{2}A \cos \tfrac{1}{2}(C+B)$$
$$= R^2-2Rr \operatorname{cosec} \tfrac{1}{2}A \sin \tfrac{1}{2}A$$
$$= R^2-2Rr;$$

$\therefore$ **required distance between centres** $= \sqrt{(R^2-2Rr)}$ ....(128)

ILLUSTRATIVE EXERCISE.

Prove that the distance between the centres of the circumcircle and the circle opposite the angle A is $\sqrt{(R^2+2Rr_1)}$.

EXAMPLES XXI.

1. Prove that, in any triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of circumscribing circle.}$$

2. If R denote the radius of the circle circumscribing ABC, show that

$$R(a^2+b^2+c^2) = abc(\cot A+\cot B+\cot C).$$

3. $r = 4R \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2} = r_1 \tan \dfrac{B}{2} \tan \dfrac{C}{2}$.

4. $r = a \sin \frac{B}{2} \sin \frac{C}{2} \sec \frac{A}{2}$.

5. If r be the radius of the circle inscribed in the triangle **ABC**, and α, β, γ the distances of the angular points from the centre of the circle, show that
$$\alpha\beta\gamma s = abcr.$$

6. Show that the squares of the distances of the angular points of a triangle from the centre of the inscribed circle, in terms of the sides of the triangle are
$$\frac{bc}{s}(s-a), \quad \frac{ca}{s}(s-b), \quad \frac{ab}{s}(s-c).$$

7. In any triangle **ABC**, join **C** to any point **D** in **AB**; let R and R_1 be the radii of the circles which circumscribe the triangles **ACD** and **BCD**, respectively. Show that $Ra = R_1 b$, and the distance between the centres of the circles is $\frac{Rc}{b}$.

8. Let the internal bisectors of the angles **A**, **B**, **C** be produced to meet the circumscribed circle again in **D**, **E**, **F**; and let $\triangle$, $\triangle'$ be the areas of the triangles **ABC** and **DEF**. Show that
$$\frac{\triangle'}{\triangle} = \frac{R}{2r}.$$

9. If **O** be the centre of the circle circumscribing the triangle **ABC**, and **CO** make angles θ and ϕ with the sides **CA** and **CB**, show that
$$c^2 \cos\theta \cos\phi = ab \sin^2 C.$$

10. Prove that the area of the triangle **ABC**
$$= 2R^2 \sin A \sin B \sin C,$$
and also
$$= r^2 \cot\frac{A}{2} \cot\frac{B}{2} \cot\frac{C}{2}.$$

11. Find the radii of the in-circle and circum-circle of the triangles whose sides are (i) $3a$, $4a$, $5a$; (ii) 4, 5, 6.

12. Find the in-radius and each of the ex-radii for the triangle with sides 13, 14, 15.

13. Compare the circum-radii of the two triangles whose sides are 13, 14, 15 and 13, 4, 15, respectively.

14. Find r_1, r_2, r_3 in the case of a triangle whose sides are 17, 10, 21.

15. If the area of a triangle is 96, and the radii of the escribed circles are 8, 12, 24, find the sides.

16. Prove that the square of the distance between the centres of the inscribed and circumscribed circles is $R^2 - 2Rr$, and hence show that r can never be greater than $\frac{1}{2}R$.

17. **ABC** is a triangle, and the inscribed circle and the escribed circle touching **BC** are drawn. Show that the points in which the former touches **AB** and **AC** are at a distance $s-a$ from **A**, and that the points in which the latter touches **AB** and **AC** produced are at a distance s from **A**, where s denotes the semi-perimeter of the triangle.

18. If r, R be the inscribed and circumscribed radii of the triangle **ABC**, and r_1, r_2, r_3 the escribed radii, prove that

$$r_1+r_2+r_3-r = 4R \quad\text{and}\quad rr_1(r_2-r_3) = (b-c)\, r_2 r_3 \tan \tfrac{1}{2}A.$$

19. $R = \dfrac{(r_1-r)(r_2-r)(r_3-r)}{4r^2}$.

20. If r_1 be the radius of the escribed circle which touches **BC** and the two other sides produced, show that

$$\frac{a^2}{rr_1} = \frac{(r_2+r_3)^2}{r_2 r_3}.$$

21. In any plane triangle, if $(a-b)(s-c) = (b-c)(s-a)$, show that the radii of the three circles, each touching one side and the other two produced, are in Arithmetical Progression.

22. Let l be the perimeter and $\triangle$ the area of the triangle **ABC**; let **D, E, F** be the centres of the escribed circles, and m the perimeter, and D the area of the triangle **DEF**. Show that

(i) $4\triangle D' = labc$;

(ii) $lm = 4D(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C)$.

23. If **O** be the centre of the inscribed circle, and if the escribed circle touch **AC** in **D** and the other sides produced in **E** and **F**, show that the triangles **DAE, OAF** are together equal to the triangle **ABC**,

24. Two tangents are drawn to a circle, making an angle of 60° with each other. In the space between them and the circle, a circle is inscribed touching both tangents and the circle. In the space between the tangents and the circle thus inscribed, another circle is inscribed, and so on continually. Prove that the area of the first circle is equal to eight times the sum of the areas of all the rest. Find the relation between the circumference of the first circle and the sum of the circumferences of all the rest.

25. If p_1, p_2, p_3 be the distances from the sides of the centre of the circum-circle, prove that

$$\frac{a}{p_1}+\frac{b}{p_2}+\frac{c}{p_3} = \frac{abc}{4p_1p_2p_3}.$$

26. Prove that the diameter of the circum-circle

$$=\sqrt[3]{\frac{abc}{\sin A \sin B \sin C}}.$$

27. $\dfrac{1}{s-a}+\dfrac{1}{s-b}+\dfrac{1}{s-c}-\dfrac{1}{s}=\dfrac{4R}{S}$.

28. $r = \dfrac{a-b}{\cot\dfrac{B}{2}-\cot\dfrac{A}{2}}$.

29. $r\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)=s.$

30. $\frac{r}{R}=4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right).$

31. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2Rr}.$ 32. $r=\frac{a\sin B\sin C}{\sin A+\sin B+\sin C}$

33. $bc\cot\frac{A}{2}+ca\cot\frac{B}{2}+ab\cot\frac{C}{2}=4Rs^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{s}\right).$

34. $\sin A+\sin B+\sin C=\frac{4Ss}{abc}.$

35. $R=\frac{1}{4}\sqrt{(b+c)^2\sec^2\frac{A}{2}+(b-c)^2\operatorname{cosec}^2\frac{A}{2}}.$

36. $S=2R^2\sin A\sin B\sin C.$ 37. $S=\frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

38. $S=Rr(\sin A+\sin B+\sin C).$ 39. $S=\frac{a^2+b^2+c^2}{4(\cot A+\cot B+\cot C)}.$

40. $abcs\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=S^2.$

41. $a\sin B\cos C+b\sin C\cos A+c\sin A\cos B=\frac{S}{abc}(a^2+b^2+c^2).$

42. $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\frac{s}{r}=4R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$

43. $S=\frac{1}{4}(a^2\cot A+b^2\cot B+c^2\cot C).$

44. $S=\frac{1}{2}a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}(\sin A)^{\frac{1}{3}}(\sin B)^{\frac{1}{3}}(\sin C)^{\frac{1}{3}}.$

45. $s^2=S\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right).$ 46. $4S=(b^2+c^2-a^2)\tan A.$

47. $S=\frac{a^2}{4}\sin 2B+\frac{b^2}{4}\sin 2A.$

48. $a^2\sin(B-C)+b^2\sin(C-A)+c^2\sin(A-B)=\frac{2S}{abc}(b-c)(c-a)(a-b).$

49. $(b+c)\tan\frac{A}{2}+(c+a)\tan\frac{B}{2}+(a+b)\tan\frac{C}{2}$
$=4R(\cos A+\cos B+\cos C).$

50. $S=\left\{\frac{abc}{8}(a\cos A+b\cos B+c\cos C)\right\}^{\frac{1}{2}}.$ 51. $r_1+r_2=c\cot\frac{C}{2}.$

52. $r_1 \cot\frac{A}{2} = r_2 \cot\frac{B}{2} = r_3 \cot\frac{C}{2} = r \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$$= 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

53. $\cot^2\frac{A}{2} = \frac{r_2+r_3}{r_1-r}$. 54. $\frac{2s}{r} + \frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = 8R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

55. $Rr_1(s-a) = Rr_2(s-b) = Rr_3(s-c) = \frac{abc}{4}$. 56. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

57. $r_1r_2+r_2r_3+r_3r_1 = s^2$. 58. $rr_1r_2r_3 = S^2$.

59. $\frac{r^3}{r_1r_2r_3} = \tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}$.

60. $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{a+b-c}{a+b+c} = \frac{r}{r_3}$. 61. $\sin\frac{A}{2} = \frac{r}{\sqrt{\{(r_2-r)(r_3-r)\}}}$.

62. $a = (r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}$.

63. $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right) = \frac{64R^3}{a^2b^2c^2}$.

64. $\frac{r_1-r}{a} + \frac{r_2-r}{b} = \frac{c}{r_3}$.

65. $a\sin B\cos B - \frac{b\sin B}{\sec(C+B)} = \frac{2rr_3}{r_2+r_1}$. 66. $\frac{1}{r_1-r} + \frac{1}{r_2+r_3} = \frac{4R}{a^2}$.

67. $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right) = \frac{16R}{r^2(a+b+c)^2}$.

68. $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{r_1+r_2+r_3}{(r_1r_2+r_2r_3+r_3r_1)^{\frac{1}{2}}}$.

69. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r} = 2\sqrt{\frac{1}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)}$.

70. $r_1^3+r_2^3+r_3^3-3r_1r_2r_3 = (4R+r)(4R+r+s\sqrt{3})(4R+r-s\sqrt{3})$.

CHAPTER XXII.

REGULAR POLYGONS AND QUADRILATERALS IN GENERAL.

238. **Geometry of regular polygons.**—DEFINITION: By a **regular** polygon is meant a polygon whose sides are all equal, and whose angles are all equal.

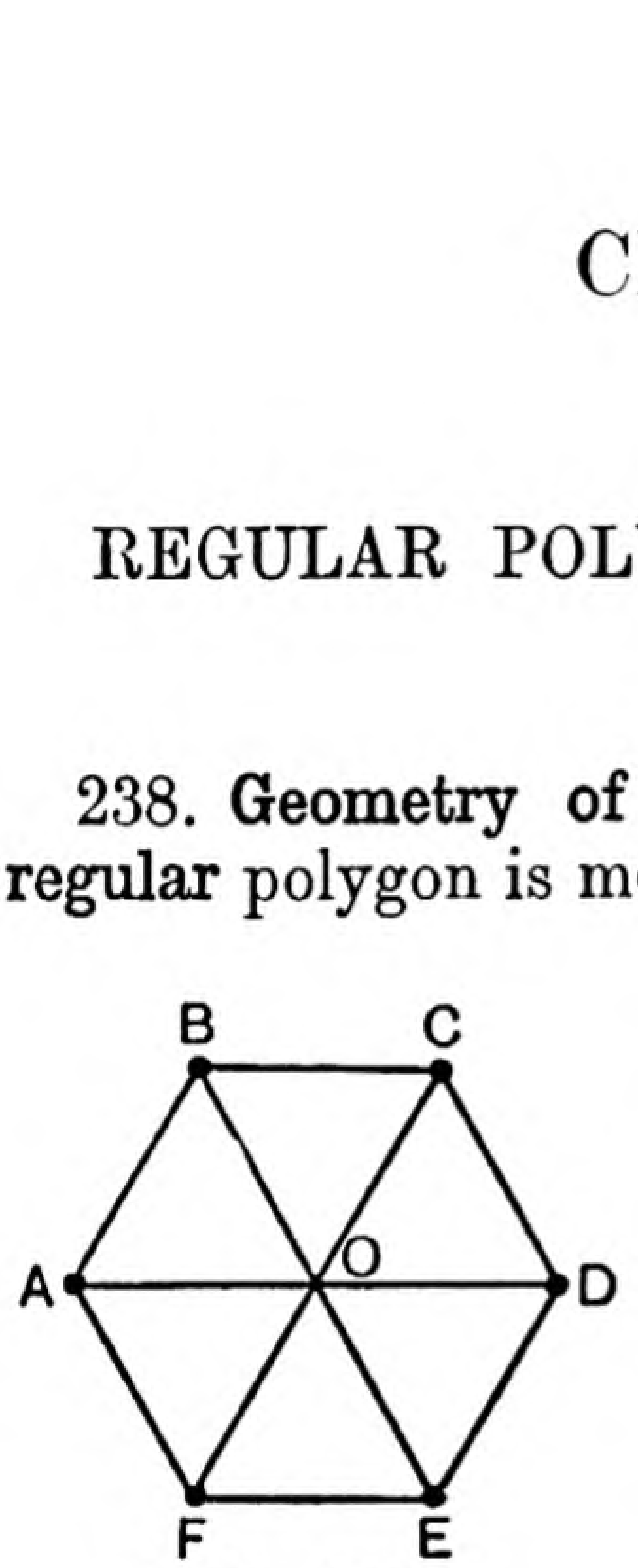


Fig. 128.

When a polygon is regular, a circle *could* be inscribed in it by the same construction as that given in Euclid IV. 13, and a circle *could* be circumscribed about it by the same construction as that given in Euclid IV. 14.

These circles are sometimes called the incircle and circumcircle, as in the case of a triangle.

To construct the circumscribing circle it is, however, sufficient to draw a circle passing through *any three* vertices of the polygon. Only one such circle can be drawn, and this must, therefore, be the required circle, and must pass through the remaining vertices of the polygon.

Similarly, to construct the inscribed circle, it is sufficient to make the circle touch three sides of the polygon and have its centre within the polygon.

From considerations of symmetry, or otherwise, it may easily be seen that the circumscribing and inscribed circles have the same centre. This point is the **centre of the polygon.**

When the number of sides is *even*, the centre is the point of intersection of the diagonals (Fig. 129).

When the number of sides is *odd*, the centre is the point of intersection of the lines each of which joins one vertex to the middle point of the opposite side or to the point of intersection of the produced sides adjacent to it. Thus, in Fig. 130, if the lines **A**a, **B**b, **C**c, . . . be drawn, they will all intersect in the centre of the pentagon, and **A**a will bisect **CD** at right angles.

Thus the inscribed and circumscribed circles can be drawn by a far simpler construction with ruler and compasses than that given in Euclid.

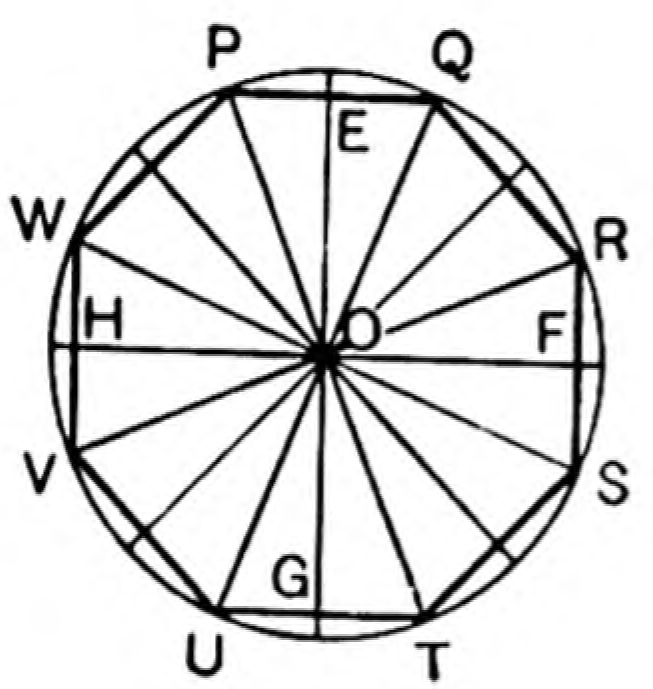


Fig. 129.

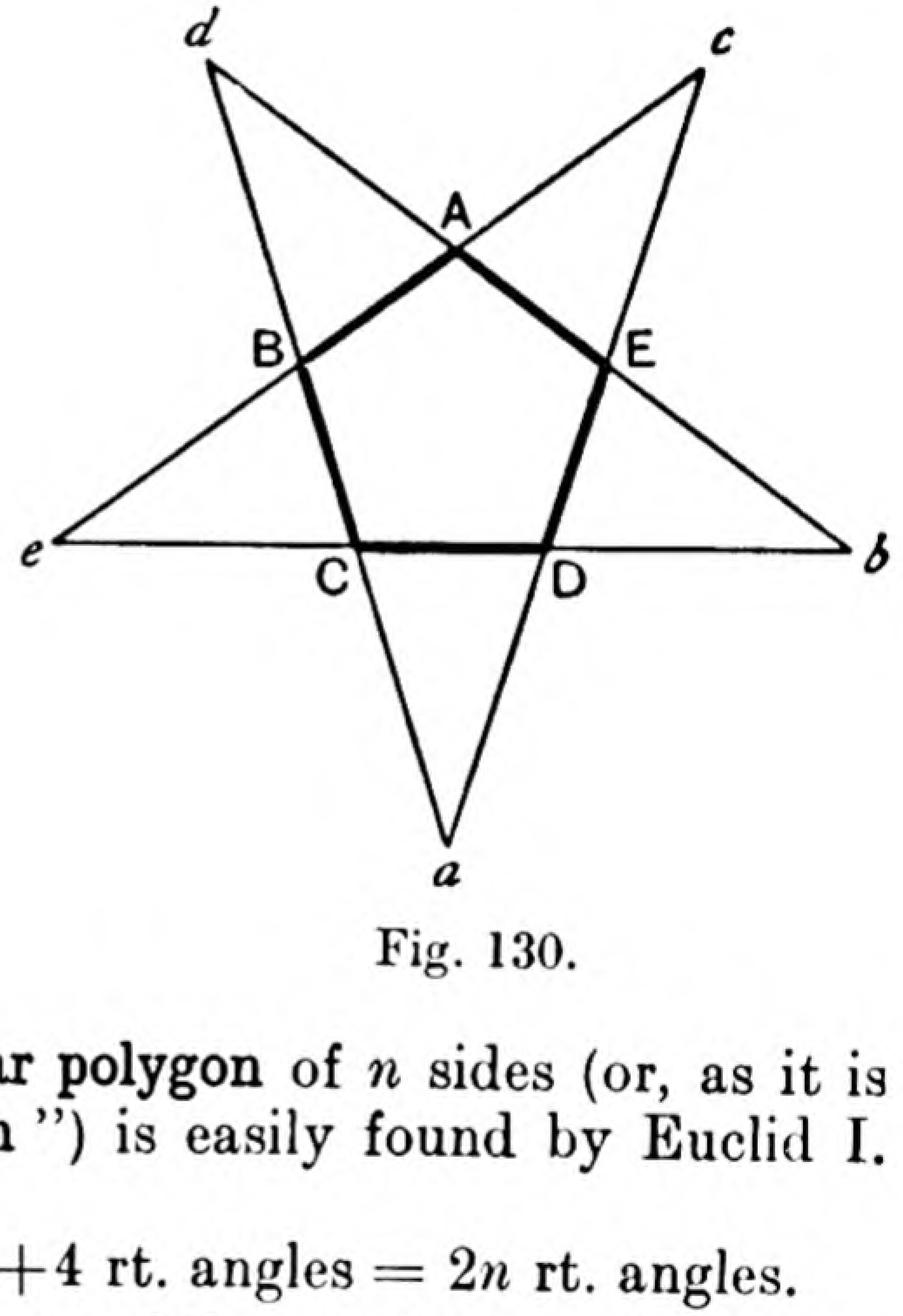


Fig. 130.

239. **The angle of a regular polygon** of n sides (or, as it is sometimes called, an "n-**gon**") is easily found by Euclid I. 32, Cor., which asserts that

sum of angles of polygon$+4$ rt. angles $= 2n$ rt. angles.

Hence, if the circular measure of the required angle be a,

$$na+2\pi = n\pi;$$

$$\therefore \quad a = \pi - \frac{2\pi}{n} \quad \text{......................}(129)$$

240. To find the radii of the circumscribed and inscribed circles of a regular polygon in terms of a side.

Let a be the length of the side of a regular polygon of n

sides, and let r, R denote the radii of the inscribed and circumscribed circles.

Let **AOB** be one of the n triangles into which the polygon can be divided by joining its centre **O** to its angular points

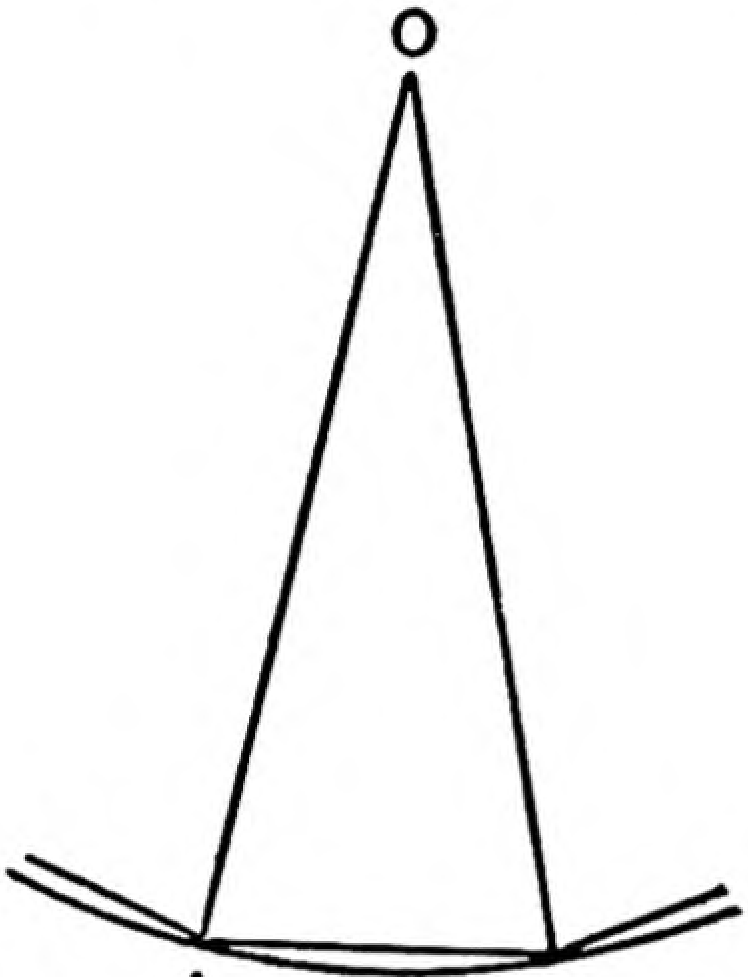


Fig. 131.

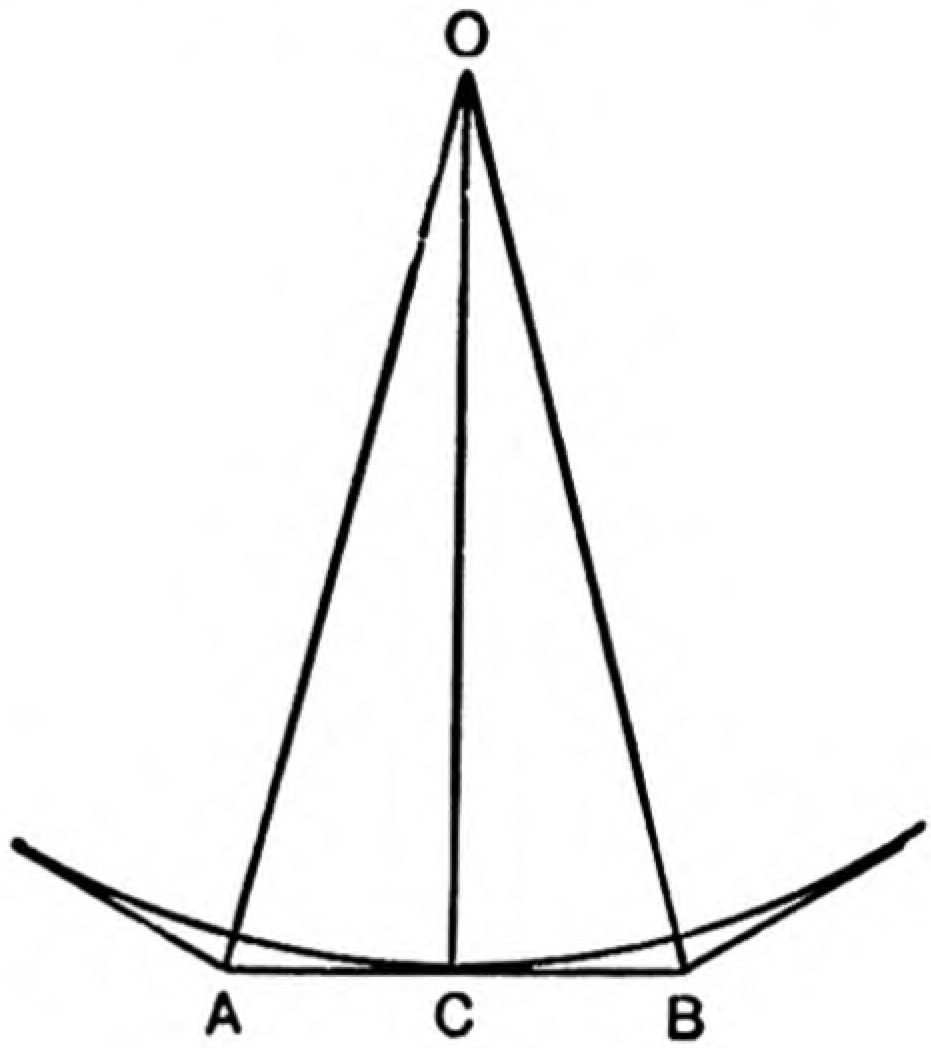


Fig. 132.

Then, $\mathbf{OA} = \mathbf{OB} = R$, the radius of the circumscribed circle (Fig. 131), and, if **C** be middle point of **AB**, **OC** is perpendicular to **AB**, and $= r$, the radius of the inscribed circle (Fig. 132).

Also, $$\mathbf{AC} = \mathbf{CB} = \frac{a}{2},$$

$$\angle \mathbf{AOB} = \frac{2\pi}{n}, \qquad \angle \mathbf{AOC} = \tfrac{1}{2}\angle \mathbf{AOB} = \frac{\pi}{n}.$$

Evidently, $$\mathbf{AC} = \frac{a}{2} = R \sin\frac{\pi}{n} = r \tan\frac{\pi}{n};$$

$$\therefore \quad R = \frac{a}{2} \operatorname{cosec}\frac{\pi}{n} \quad \text{.......................(130)}$$

$$\therefore \quad r = \frac{a}{2} \cot\frac{\pi}{n} \quad \text{.........................(131)}$$

Cor.—Perimeter $= na = 2nR \sin\frac{\pi}{n} = 2nr \tan\frac{\pi}{n}$.

241. To find the area of the polygon.

Let **ABCD** ... be the polygon of n sides, **O** the centre, **P** the middle point of **BC**.

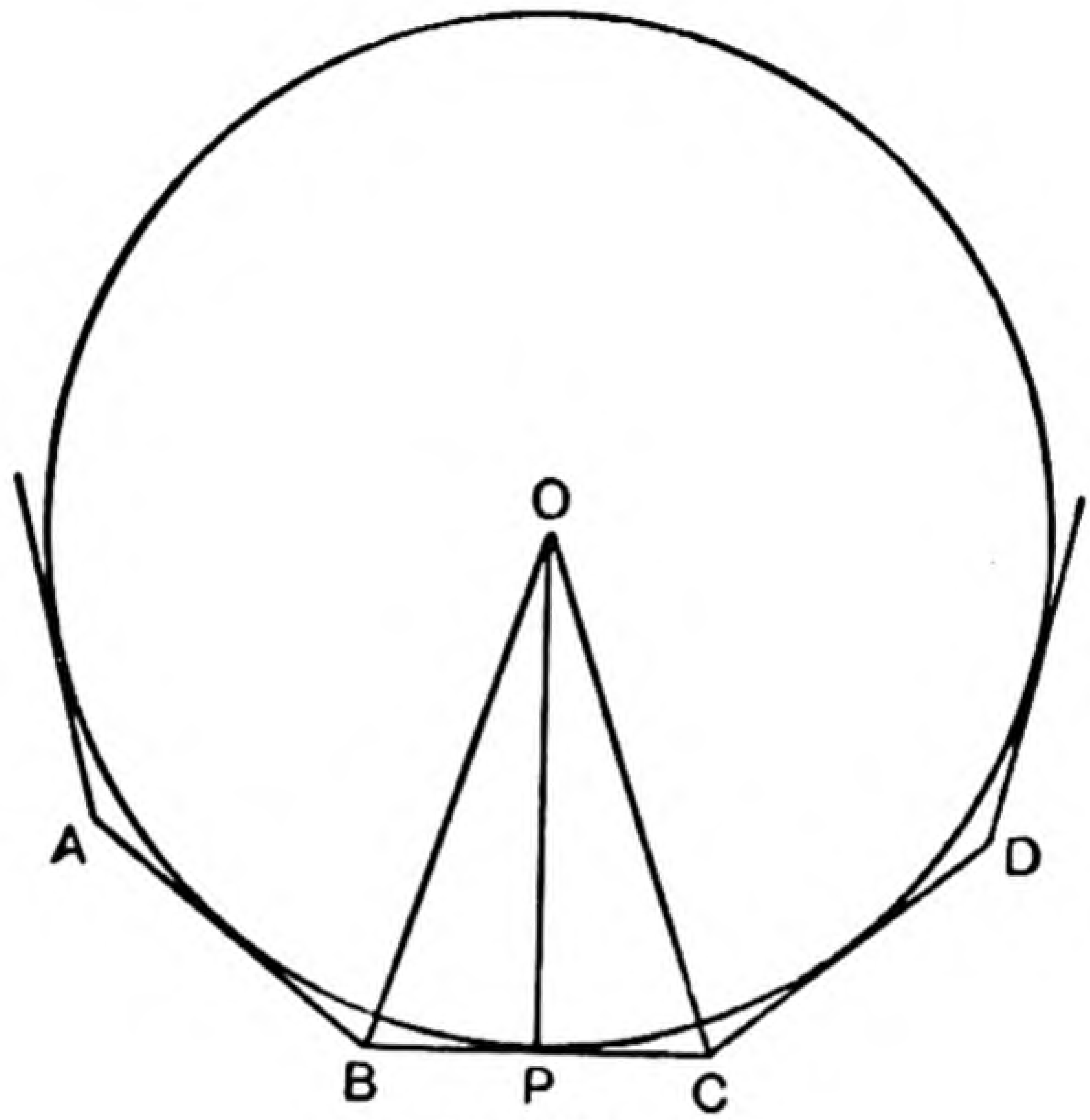


Fig. 133.

Then area of triangle $\mathbf{OBC} = \frac{1}{2}\mathbf{OP}.\mathbf{BC} = \mathbf{OP}.\mathbf{PC}$,

and polygon $= n \times$ triangle,

$$= \tfrac{1}{2}n.\mathbf{OP}.\mathbf{BC} = n.\mathbf{OP}.\mathbf{PC} = \tfrac{1}{2}nra.$$

In terms of the side a,

$$\mathbf{OP} = \frac{a}{2}\cot\frac{\pi}{n} \quad \text{and} \quad \mathbf{PC} = \frac{a}{2};$$

$$\therefore \quad \text{area} = \frac{na^2}{4}\cot\frac{\pi}{n}.$$

In terms of the "in-radius" r,

$$\mathbf{PC} = r\tan\frac{\pi}{n} \quad \text{and} \quad \mathbf{OP} = r$$

$$\therefore \quad \text{area} = nr^2\tan\frac{\pi}{n}.$$

In terms of the "circum-radius" R,

$$\mathbf{PC} = R \sin\frac{\pi}{n} \quad \text{and} \quad \mathbf{OP} = R\cos\frac{\pi}{n};$$

$$\therefore \quad \text{area} = nR^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{nR^2}{2}\sin\frac{2\pi}{n}$$

[which can also be deduced from the property that

$$\text{area} = n.\text{area } \mathbf{AOB} = n.\tfrac{1}{2}\mathbf{OA}.\mathbf{OB}\sin \mathbf{AOB}],$$

Hence the area of a regular polygon of n sides each of length a

$$= \frac{nar}{2} = \frac{na^2}{4}\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n} = n\frac{R^2}{2}\sin\frac{2\pi}{n} \quad \ldots.(132)$$

Cor.—Since area $= n.\mathbf{OP}.\mathbf{PC} = \frac{1}{2}n.\mathbf{OP}.\mathbf{BC}$;

$$\therefore \quad \text{area} = \tfrac{1}{2}\text{ perimeter} \times r.$$

Hence, if S denote the area and s the semiperimeter, we have

$$r = \frac{S}{s},$$

just as in the case of a triangle.

Ex. To find the areas of the two regular pentagons inscribed in and circumscribed about a circle of radius 1 metre.

Here π/n or half the angle subtended by a side at the centre

$$= 180°/5 = 36°.$$

Hence the areas S_1, S_2 of the inscribed and circumscribing pentagons are $S_1 = \frac{1}{2}.5.1^2 \sin 72°$ and $S_2 = 5.1^2 \tan 36°$ (sq. metres) respectively.

From the value of sin 18°, viz. $\frac{1}{4}(\sqrt{5}-1)$ or otherwise, we have

$$\sin 72° = \cos 18° = \tfrac{1}{4}\sqrt{(10+2\sqrt{5})}$$

$$\tan 36° = \sqrt{\frac{1-\cos 72°}{1+\cos 72°}} = \sqrt{\frac{1-\sin 18°}{1+\sin 18°}} = \sqrt{\frac{4-\sqrt{5}+1}{4+\sqrt{5}-1}}$$

$$= \sqrt{\frac{5-\sqrt{5}}{3+\sqrt{5}}} = \sqrt{\frac{(5-\sqrt{5})(3-\sqrt{5})}{9-5}} = \sqrt{\frac{20-8\sqrt{5}}{4}}$$

$$= \sqrt{(5-2\sqrt{5})};$$

$\therefore \quad S_1 = \dfrac{5\sqrt{(10+2\sqrt{5})}}{8}$ and $S_2 = 5\sqrt{(5-2\sqrt{5})}$ sq. metres, or, approximately, $S_1 = 2{\cdot}36$ sq. metres, $S_2 = 3{\cdot}67$ sq. metres.

242. To find the area of any quadrilateral in terms of its sides and the sum of two opposite angles.

Let a, b, c, d denote the sides **AB, BC, CD, DA,** respectively, of the quadrilateral **ABCD.** Join **BD.** Let the area of the quadrilateral $= S$, and the semi-sum of the angles A and $C = \theta$.

Then $$\text{area } \mathbf{ABCD} = \triangle\mathbf{ABD} + \triangle\mathbf{BCD};$$

$$\therefore\quad S = \tfrac{1}{2}ad\sin A + \tfrac{1}{2}bc\sin C \quad\ldots\ldots\ldots\ldots\text{(i)}$$

Again, $$\mathbf{BD}^2 = a^2+d^2-2ad\cos A = b^2+c^2-2bc\cos C;$$

$$\therefore\quad a^2+d^2-b^2-c^2 = 2\,(ad\cos A - bc\cos C) \quad\ldots\ldots\ldots\ldots\text{(ii)}$$

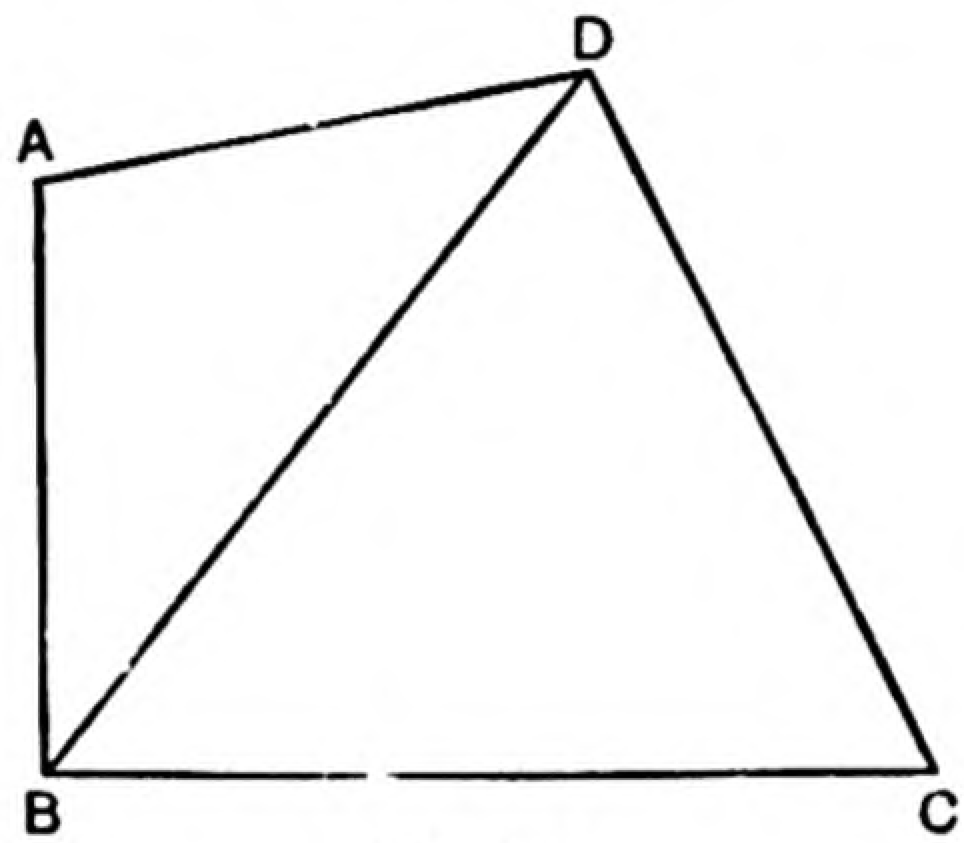


Fig. 134.

Squaring (i) and (ii), we have

$$4\,(a^2d^2\sin^2 A + 2abcd\sin A\sin C + b^2c^2\sin^2 C) = 16S^2,$$

$$4\,(a^2d^2\cos^2 A - 2abcd\cos A\cos C + b^2c^2\cos^2 C) = (a^2+d^2-b^2-c^2)^2.$$

Adding the last two equations, we get

$$4\{a^2d^2 - 2abcd\,(\cos A\cos C - \sin A\sin C) + b^2c^2\}$$
$$= 16S^2 + (a^2+d^2-b^2-c^2)^2;$$

$$\begin{aligned}\therefore\quad 16S^2 &= 4a^2d^2+4b^2c^2-(a^2+d^2-b^2-c^2)^2-8abcd\cos(A+C)\\
&= 4a^2d^2+4b^2c^2-(a^2+d^2-b^2-c^2)^2-8abcd\cos 2\theta\\
&= 4a^2d^2+4b^2c^2+8abcd-(a^2+d^2-b^2-c^2)^2-16abcd\cos^2\theta\\
&= \{(2ad+2bc)^2-(a^2+d^2-b^2-c^2)^2\}-16abcd\cos^2\theta\\
&= \{(2ad+a^2+d^2)-(b^2+c^2-2bc)\}\{(b^2+c^2+2bc)\\
&\qquad -(a^2+d^2-2ad)\}-16abcd\cos^2\theta\\
&= \{(a+d)^2-(b-c)^2\}\{(b+c)^2-(a-d)^2\}-16abcd\cos^2\theta\\
&= (a+b-c+d)(a-b+c+d)(b+c+d-a)(a+b+c-d)\\
&\qquad -16abcd\cos^2\theta.\end{aligned}$$

Let $a+b+c+d = 2s$;

then $b+c+d-a = 2(s-a)$, $a-b+c+d = 2(s-b)$,

$a+b-c+d = 2(s-c)$, $a+b+c-d = 2(s-d)$;

$\therefore \quad 16S^2 = 16(s-a)(s-b)(s-c)(s-d)-16abcd\cos^2\theta$;

$$\therefore \quad \boldsymbol{S = \{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\theta\}^{\frac{1}{2}}} \quad \text{.......(133)}$$

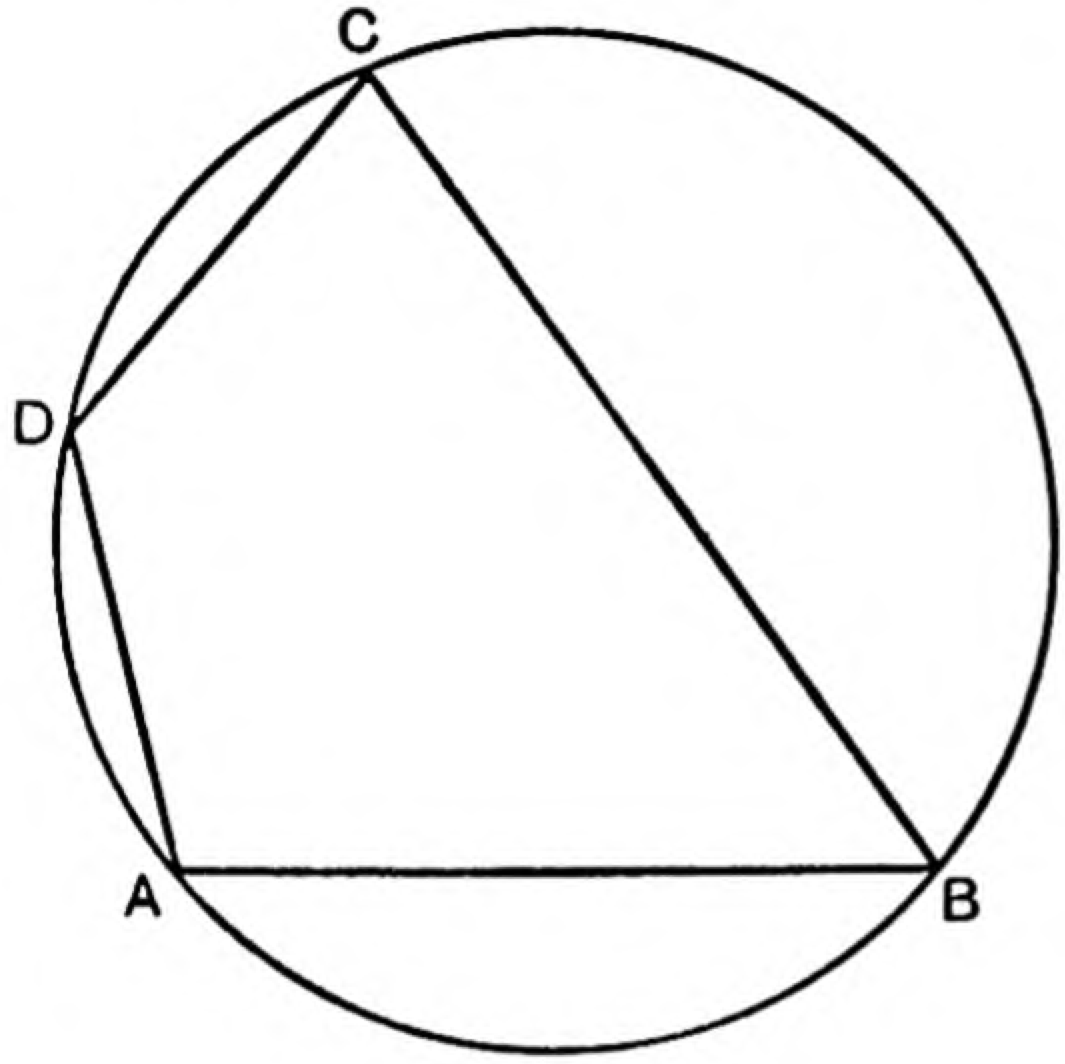


Fig. 135.

Cor. 1.—*Area of a quadrilateral inscribed in a circle.*

Since the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles (Euc. III. 22), we have $2\theta = \pi$;

$$\therefore \quad \cos\theta = \cos\frac{\pi}{2} = 0,$$

and

$$\text{area} = \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}.$$

Cor. 2.—*Area of a quadrilateral circumscribed about a circle.*

Let the sides **AB, BC, CD, DA** touch the circle at the points **E, F, G, H**, respectively.

Then

AE = **AH**,

BE = **BF**,

CF = **CG**,

DG = **DH**;

$\therefore$ $2s = 2$ (**AH** + **HD** + **BF** + **FC**) $= 2(b+d)$,

and similarly

$= 2(a+c)$;

$\therefore \quad s = b+d = a+c$;

$\therefore \quad s-a = c$,

$s-b = d$,

$s-c = a$,

$s-d = b$;

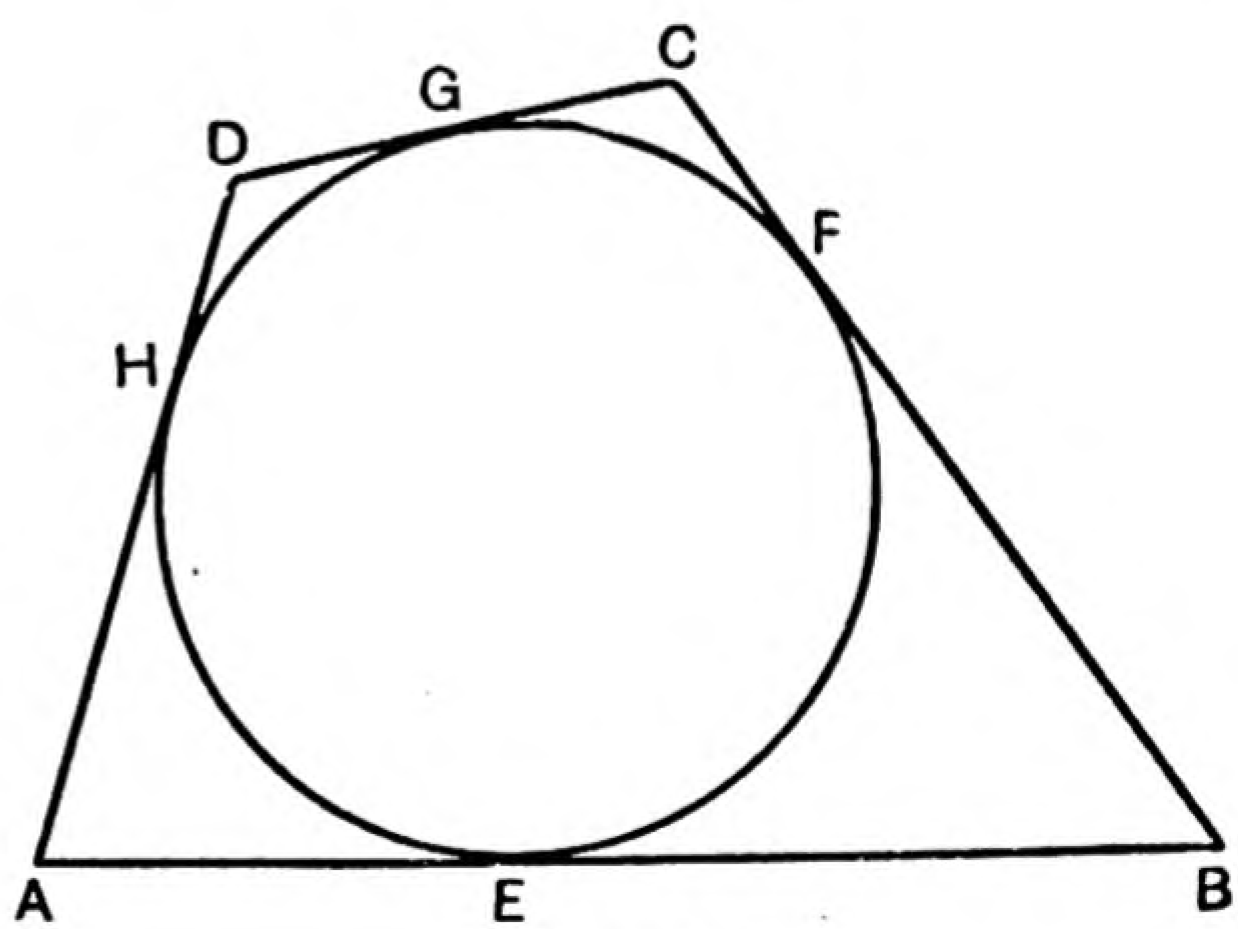


Fig. 136.

$$\therefore \quad \text{area} = \sqrt{\{abcd - abcd\cos^2\theta\}} = \sqrt{\{abcd\sin^2\theta\}}$$

$$= \sqrt{(abcd)}\sin\theta \quad \text{..............................(133B)}$$

***243.** *If x, y are the lengths of the diagonals of a quadrilateral and 2θ the sum of a pair of opposite angles, to prove that*

$$x^2y^2 = a^2c^2+b^2d^2-2abcd\cos 2\theta.$$

Let **ABCD** be the quadrilateral, and let $\mathbf{AC} = x$, $\mathbf{BD} = y$. Make

$$\angle \mathbf{ABE} = \angle \mathbf{DBC}$$

and $$\angle \mathbf{BAE} = \angle \mathbf{BDC}.$$

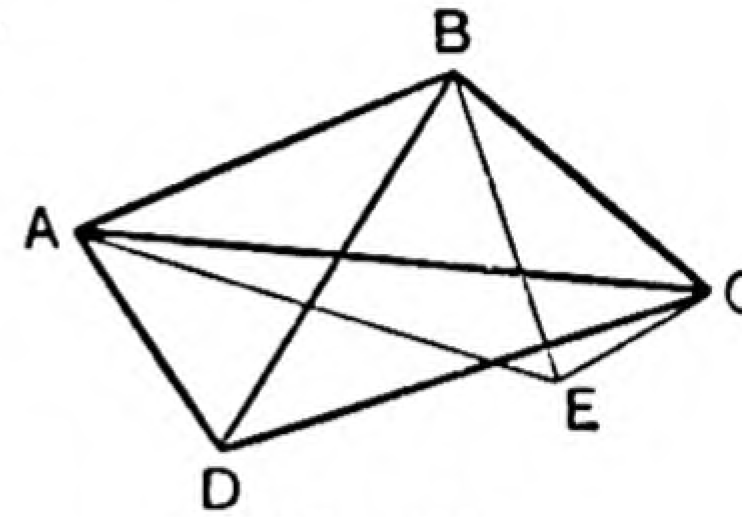


Fig. 137.

Then the $\triangle$s **BAE, BDC** are similar;

$$\therefore \quad \frac{\mathbf{AB}}{\mathbf{BE}} = \frac{\mathbf{DB}}{\mathbf{BC}}$$

and $$\frac{\mathbf{AE}}{\mathbf{AB}} = \frac{\mathbf{DC}}{\mathbf{DB}} = \frac{c}{y};$$

$$\therefore \quad \mathbf{AE} = \frac{ac}{y}, \text{ also } \angle \mathbf{BEA} = \angle \mathbf{BCD}.$$

Again, $\angle \mathbf{CBE} = \angle \mathbf{DBA}$, and **BC, BE** are proportional to **BD, BA**, therefore the $\triangle$s **BCE, BDA** are similar;

$$\therefore \quad \frac{\mathbf{EC}}{\mathbf{CB}} = \frac{\mathbf{AD}}{\mathbf{DB}} = \frac{d}{y};$$

$$\therefore \quad \mathbf{EC} = \frac{bd}{y}, \text{ also } \angle \mathbf{BEC} = \angle \mathbf{BAD}.$$

Hence $$\angle \mathbf{AEC} = \angle \mathbf{BCD} + \angle \mathbf{BAD} = 2\theta,$$

and $$x^2 = \mathbf{AC}^2 = \mathbf{AE}^2+\mathbf{EC}^2-2\mathbf{AE}.\mathbf{EC}\cos \mathbf{AEC}.$$

$$= \frac{a^2c^2}{y^2} + \frac{b^2d^2}{y^2} - 2\frac{ac}{y}\frac{bd}{y}\cos 2\theta;$$

$$\therefore \quad x^2y^2 = a^2c^2+b^2d^2-2abcd\cos 2\theta.$$

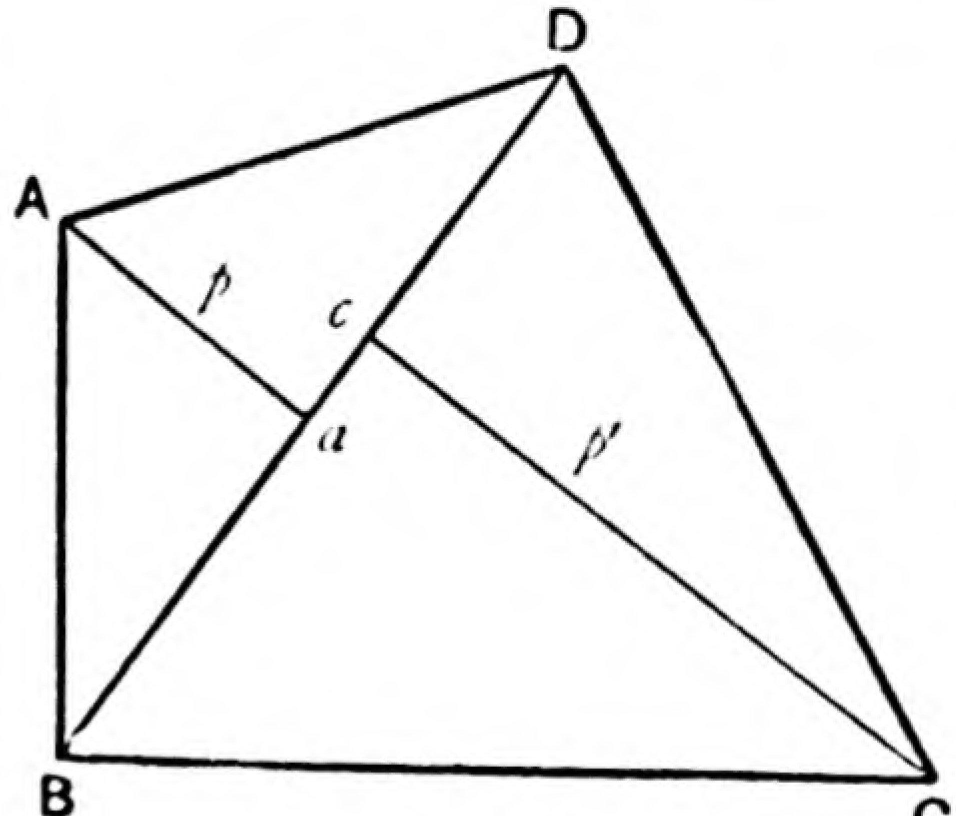


Fig. 138.

Ex. Prove that, if w denote the angle between the diagonals of the quadrilateral,

$$2xy\cos w = b^2+d^2-a^2-c^2.$$

Drop $\mathbf{A}a$, $\mathbf{C}c$ perpendicular on **BD**.

Then it is easy to see that

$$x\cos w = ca = \mathbf{B}c - \mathbf{B}a$$

$$= \mathbf{BC}\cos\mathbf{DBC} - \mathbf{BA}\cos\mathbf{DBA}$$

$$= b.\frac{b^2+y^2-c^2}{2by} - a.\frac{a^2+y^2-d^2}{2ay};$$

$$\therefore \quad 2xy\cos w = b^2+d^2-a^2-c^2.$$

Illustrative Exercises.

(1) Prove that the area of the quadrilateral $= \frac{1}{2}xy \sin w$.

(2) Hence, deduce the result of § 242 from the expressions for x^2y^2 and $x^2y^2 \cos^2 w$, obtained in § 243 and the above example.

EXAMPLES XXII.

1. Two regular polygons have the number of their sides in the ratio 2 : 3, and their angles are in the ratio 3 : 4. Find the number of sides.

2. Find the radius of the circumscribed circle of a regular hexagon, side 8 ft.

3. Find the radius of the circumscribed circle of a regular nonagon whose inscribed radius is 3 ft. long.

4. If R and r be the radii of the circum-circle and in-circle of a regular polygon of n sides, each $= a$, prove that

$$R+r = \frac{a}{2}\cot\frac{\pi}{2n}.$$

5. Find the areas of the following regular polygons:—

(i) Pentagon, side 3 ft.; (iv) Octagon, side, 8 ft.;
(ii) Hexagon, side 10 ft.; (v) Nonagon, side 12 ft.
(iii) Heptagon, side 9 ft.;

6. If a regular pentagon and a regular decagon have the same peri meter, find the ratio of their areas.

7. Find the area of a regular polygon of n sides of given perimeter p.

8. If regular octagons be described about and in the same circle, find the ratio of their areas.

9. If a quadrilateral be formed of four rods jointed at the corners, show that it will include the greatest area when it can be inscribed in a circle.

10. If the sides of a quadrilateral be 23, 29, 37, and 41 ft., find its greatest area.

11. Prove that, if a quadrilateral be inscribed in one circle and circumscribed about another, its area is $\sqrt{abcd}$.

12. If a quadrilateral be circumscribed about a circle, prove that it area is least when it can be also inscribed in another circle.

13. If **ABCD** be a quadrilateral inscribed in a circle, prove that

$$\mathbf{AC}^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

14. If **ABCD** be a quadrilateral inscribed in a circle, prove that

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-b)}{(s-c)(s-d)}}.$$

15. If **ABCD** be a quadrilateral which can be both inscribed in a circle and circumscribed about one, prove that

$$\tan^2\frac{A}{2}=\frac{bc}{ad}\quad\text{and}\quad \tan^2\frac{D}{2}=\frac{ab}{cd}.$$

16. If ϕ be the angle between the diagonals of a quadrilateral **ABCD**, its area is $\frac{1}{4}(a^2-b^2+c^2-d^2)\tan\phi.$

17. If the lines joining mid-points of opposite sides of a quadrilateral **ABCD** be equal, then

$$a^2+c^2=b^2+d^2.$$

18. If x be the length of the diagonal **AC** of the quadrilateral **ABCD**, prove that

$$\{x^2(ab+cd)-(ac+bd)(ad+bc)\}^2$$
$$=4abcd\cos^2\omega\{(x^2-a^2-b^2)(x^2-c^2-d^2)+4abcd\sin^2\omega\},$$

ω being the semi-sum of the opposite angles.

19. **ABCD** is a quadrilateral whose diagonals **AC**, **BD** intersect in **E**. The angle **AEB** is $105°\,20'$, and **AC** and **BD** are severally 343·64 and 673·75 ft. long. Find the number of square feet in the quadrilateral.

20. Find a formula for the area of a rectangle when the length of the diagonals and the angle between them is given.

If the diagonal of a rectangle be 638·64 ft. long, and the angle between the diagonals be $106°\,9'$, calculate the area of the rectangle.

21. The difference between the areas of the hexagon and pentagon circumscribed about a circle is 5 sq. ft. Show that the square of the radius can be found in the form $\dfrac{\cos\theta\cos 36°}{\sin(36°-\theta)}$, and calculate θ and the length of the radius of the circle.

22. A circle can be inscribed in a quadrilateral, three of whose sides taken in order are 5, 4, 7, and the quadrilateral itself is inscribed in a circle. Show that the sine of the angle between the diagonals is $\dfrac{8\sqrt{70}}{67}$. As this is nearly unity, can it be safely asserted that the angle is nearly a right angle? Give reasons for your answer.

23. **ABCD** is a quadrilateral inscribed in a circle, and the sides **AB**, **BC**, **CD**, **DA** are denoted by a, b, c, d. Prove that

$$(s-b)\tan\frac{A}{2}=(s-d)\tan\frac{B}{2},$$

where s denotes the semi-perimeter.

24. The difference between the perimeters of an inscribed and a circumscribed regular dodecagon equals a. Show that the difference between their areas equals

$$\frac{a^2}{192\left(1-\cos\frac{\pi}{12}\right)^2}.$$

CHAPTER XXIII.

LIMITS OF TRIGONOMETRIC FUNCTIONS.

244. In § 10 we defined the **length of the circumference of a circle** as the limit of the perimeter of an inscribed polygon when its number of sides is indefinitely increased. This definition assumes that such a limit exists, and is finite. Moreover, in the present chapter we shall make use of the fact that the limit is the same for a circumscribing as for an inscribed polygon.

These properties may be investigated as follows:—

First let p_n, p_{2n} be the perimeters of two polygons of n and $2n$ sides respectively, inscribed in a circle of radius r. Then this circle is the circum-circle of the polygons;

$$\therefore \quad p_n = 2nr \sin\frac{\pi}{n} \text{ and } \therefore \quad p_{2n} = 4nr\sin\frac{\pi}{2n}$$

(the latter being got by writing $2n$ for n in the former);

$$\therefore \quad p_n = p_{2n}\cos\frac{\pi}{2n}.$$

But $\qquad \cos\pi/2n < 1; \qquad \therefore \quad p_n < p_{2n}.$

Hence, (i) *if the number of sides be repeatedly doubled, the perimeter of the inscribed polygon continually increases.*

Again, let P_n, P_{2n} be the perimeters of two polygons of n and $2n$ sides circumscribing the same circle. Then

$$P_n = 2nr\tan\frac{\pi}{n}, \quad P_{2n} = 4nr\tan\frac{\pi}{2n};$$

$$\therefore \quad P_n = \frac{P_{2n}}{1-\tan^2\dfrac{\pi}{2n}}; \text{ whence } P_n > P_{2n}.$$

Hence, (ii) *if the number of sides be repeatedly doubled the perimeter of the circumscribed polygon continually decreases.*

Lastly, $$p_n : P_n = \sin\frac{\pi}{n} : \tan\frac{\pi}{n} = \cos\frac{\pi}{n} : 1.$$

When n is made infinitely great,

$$\cos\frac{\pi}{n} = \cos\frac{\pi}{\infty} = \cos 0 = 1 \quad \text{and} \quad \therefore \quad p_n = P_n.$$

Hence, (iii) *when the number of sides is made indefinitely great, the perimeters of the inscribed and circumscribing polygons become equal.*

If we imagine the number of sides to be increased by repeatedly doubling, it follows from result (i) that the common final limit is greater than the perimeter of any inscribed polygon of the series, and from (ii) that this limit is less than the perimeter of any circumscribing polygon. Hence it must be finite, since it lies between these finite values.

COR.—If $\cos \pi/n$ be known, $\cos \pi/2n$ may be calculated from the formula $\cos\frac{1}{2}A = \sqrt{\{\frac{1}{2}(1+\cos A)\}}$. Hence, starting with a polygon of known perimeter, the successive perimeters $p_{2n}, p_{4n}, p_{8n}, \ldots$ may be calculated numerically and the circumference thus found; this method is a simplification of that given in § 9.

245. To find the area of a circle of radius r.

Let two regular polygons of n sides be respectively inscribed and circumscribed about the circle. Then the circle is evidently larger than the inscribed, and smaller than the circumscribed, polygon. Now, if S_1, S_2 be the areas of these,

$$S_1 = \frac{nr^2}{2}\sin\frac{2\pi}{n} = nr^2 \sin\frac{\pi}{n}\cos\frac{\pi}{n} \text{ and } S_2 = nr^2\tan\frac{\pi}{n};$$

$$\therefore \quad S_1 = S_2\cos^2\frac{\pi}{n}.$$

But, if n be made infinitely large,

$$\cos\pi/n = 1, \quad \text{and} \quad \therefore \quad S_1 = S_2.$$

Therefore in the limit each must be equal to the area of the circle which is intermediate between them.

Now, if s_2 is the semi-perimeter of the circumscribing polygon, then, by § 241, Cor.,

$$S_2 = rs_2.$$

We have just proved that the limit of S_2 is the area of the circle. Also the limit of the perimeter $2s_2$ is the circumference of the circle (by § 244);

$$\begin{aligned} \therefore \quad \textbf{area of circle} &= r \times \tfrac{1}{2}\text{ circumference} \\ &= r \times \tfrac{1}{2}(2\pi r), \quad \text{by definition of } \pi, \\ &= \boldsymbol{\pi r^2} \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(134)} \end{aligned}$$

COR. *The area of a circle is equal to the area of a triangle whose base is the circumference, and altitude the radius, of the circle.*

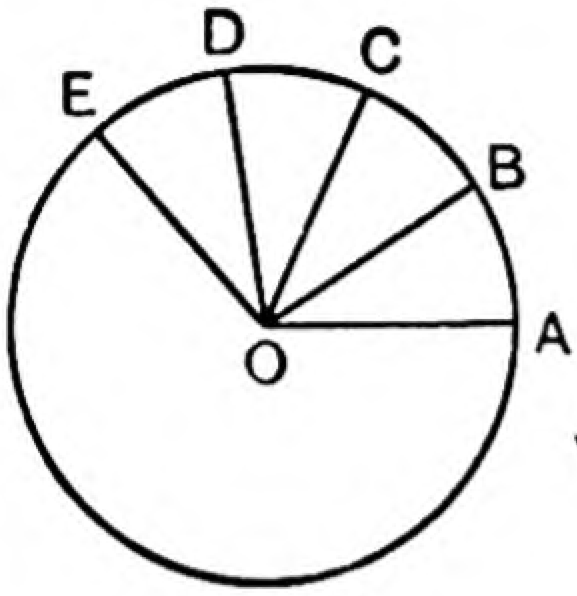


Fig. 139.

246. In finding the area of a sector of a circle we shall assume that sectors of a circle are proportional to the arcs they subtend at the centre. This may be proved as follows:—

Let **AOB, BOC, COD, DOE,** be a number of sectors subtending equal angles θ at the centre.

Then, if the sector **AOB** be cut out, it can be superposed on any other sectors, say **DOE,** so that **OA** coincides with **OD,** and **OB** with **OE,** and the arc **AB** will then coincide with **DE,** since both have the same centre and radius.

Hence the sector **AOB** is equal to the sector **DOE,** and similarly to each of the other sectors. Hence the sector **AOC,** which subtends an angle 2θ, is twice the sector **AOB,** subtending θ. Similarly, the sectors **AOD, AOE,** which subtend 3θ and 4θ, are 3 and 4 times the sector **AOB,** and so on.

Thus the sector is in every case proportional to the angle it subtends at the centre.

247. To find the area of a sector of a circle.

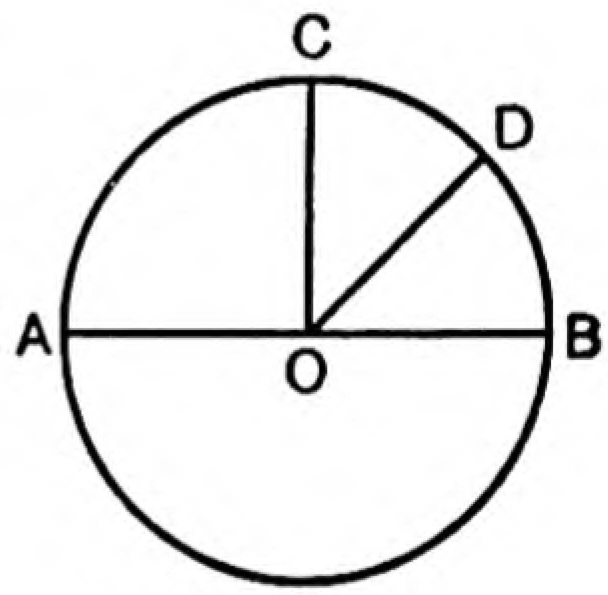


Fig. 140.

Let **BOD** be the sector, and let it subtend at **O** an angle **BOD** $= \theta$ radians.

Produce **BO** to **A.** Then the semicircle **BCA** subtends at **O** an angle of two right angles, or π radians.

Since the sector and semicircle are proportional to the angles they subtend at **O,**

$\therefore$ sector **BOD** : semicircle **BCA** $=$ $\angle$**BOD** : 2 rt. angles
$= \theta : \pi.$

But area of semicircle **BCA** $= \frac{1}{2}\pi r^2$;

$\therefore$ **area of sector** $= \frac{1}{2}\theta r^2$ ....**(135)**

If the angle of the sector be *A degrees*, its circular measure is $\pi A/180°$, and hence its area is

$$= \frac{\pi A r^2}{360}.$$

248. To find the area of a segment of a circle.

The segment **ACB** intercepted between the chord **AB** and the arc **ACB** may be regarded as the difference between the sector **AOCB** and $\triangle$ **AOB**. Drop **BE** perpendicular on **OA**.

Let

θ = circular measure of $\angle$ **BOA**.

Then

area of sector **OBCA** $= \frac{1}{2}r^2\theta$,

area of $\triangle$ **AOB** $= \frac{1}{2}$**OA.EB**

$= \frac{1}{2}r \cdot r \sin\theta$

$= \frac{1}{2}r^2 \sin\theta$;

$\therefore$ **area of segment**

$= \frac{1}{2}r^2(\theta - \sin\theta)$ ......(136)

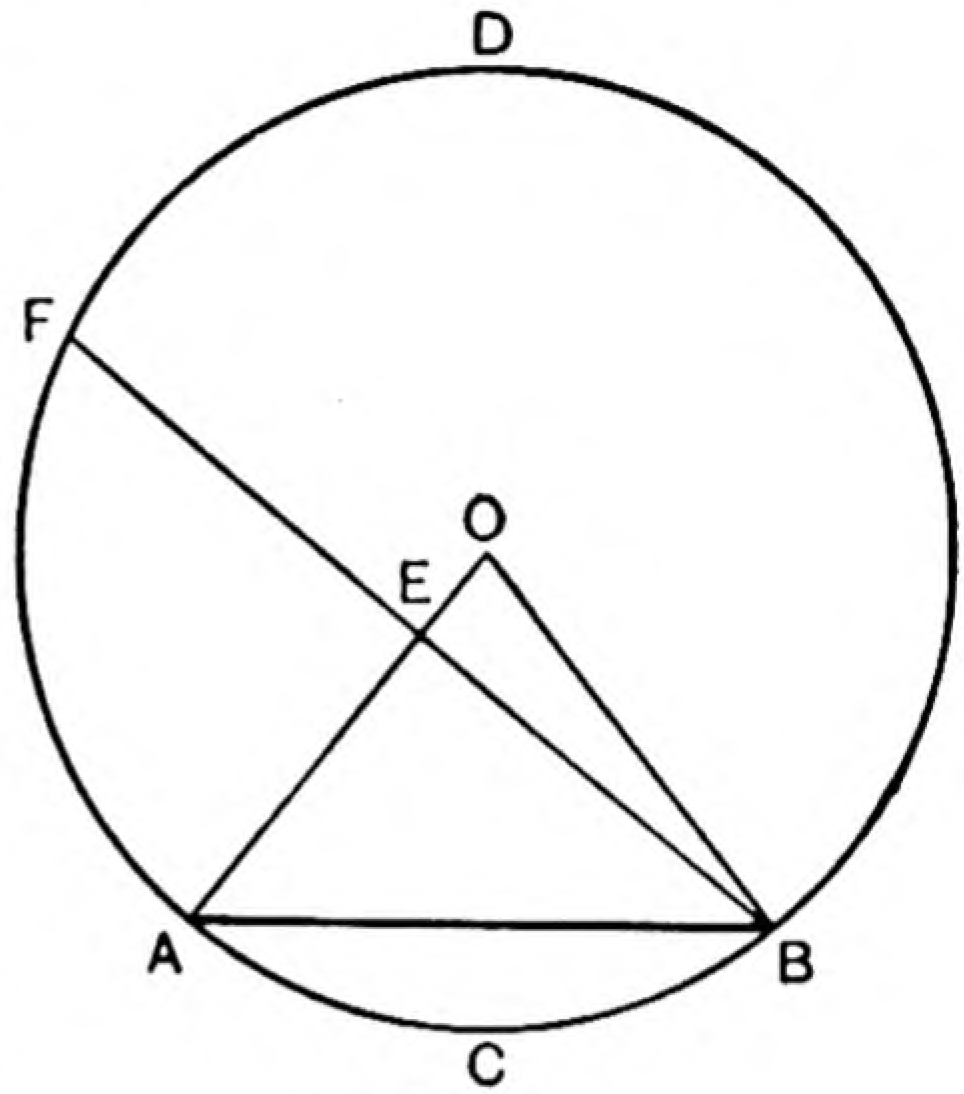


Fig. 141.

If the segment is greater than a semicircle, as **ABDF**, the same result holds; for, if the whole angle subtended at **O** be θ, the angle **AOB** subtended by the chord **AB** is $2\pi - \theta$, and

area of segment = sector **OBDA** + $\triangle$ **AOB** $= \frac{1}{2}r^2\{\theta + \sin(2\pi - \theta)\}$

$= \frac{1}{2}r^2(\theta - \sin\theta)$, as before ....(136)

249. To prove that the sine, the circular measure, and the tangent of an acute angle are in ascending order of magnitude.

We have to prove that, if θ be the circular measure of an angle $< \frac{1}{2}\pi$, then

$$\sin\theta < \theta \quad \text{and} \quad \theta < \tan\theta;$$

or, as we may write these inequalities,

$$\sin\theta < \theta < \tan\theta.$$

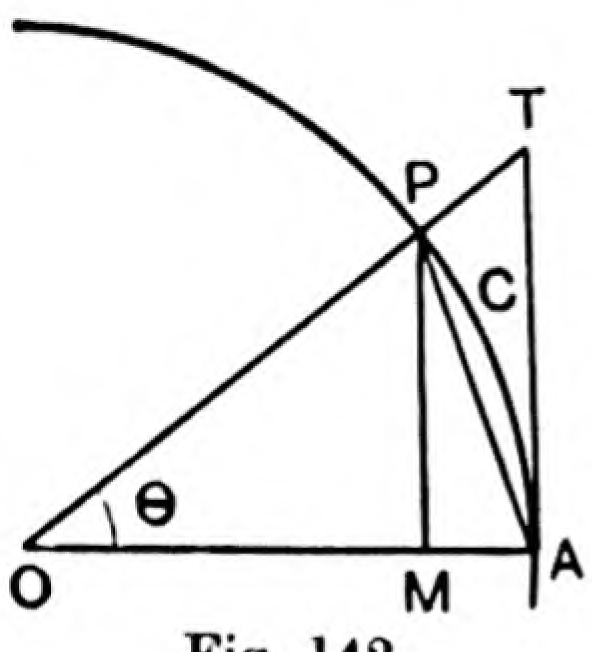


Fig. 142.

Let $\angle$ **AOP** $= \theta$. About **O** as centre describe a circular arc **AP** with any radius r. Draw the perpendicular **PM** and tangent **AT**. Then the areas

$\triangle$ **OAP**, sector **AOPC**, and $\triangle$ **OAT**

are in ascending order of magnitude. But the measures of these areas are respectively equal to

$$\tfrac{1}{2}\mathbf{OA}\times\mathbf{MP},\quad \tfrac{1}{2}\mathbf{OA}\times\text{arc }\mathbf{AP},\quad\text{and}\quad \tfrac{1}{2}\mathbf{OA}\times\mathbf{AT};$$

$$\therefore\quad \mathbf{MP} < \text{arc }\mathbf{AP} < \mathbf{AT};$$

$$\therefore\quad \frac{\mathbf{MP}}{r} < \frac{\text{arc }\mathbf{AP}}{r} < \frac{\mathbf{AT}}{r};$$

But arc $\mathbf{AT} \div r =$ circular measure of $\angle\mathbf{AOP} = \theta$;

$$\therefore\quad \sin\theta < \theta < \tan\theta \quad \text{.....................(137)}$$

In the above proof we have assumed that the area of the sector **AOPC** $= \frac{1}{2}r\times$ arc **AP**; this has been proved independently in § 247.

250. If the angle whose circular measure is θ be made infinitely small, to prove that

$$\frac{\theta}{\sin\theta} = 1 \quad\text{and}\quad \frac{\theta}{\tan\theta} = 1.$$

Starting with the property that θ lies between $\sin\theta$ and $\tan\theta$, divide each of these by $\sin\theta$;

$$\therefore\quad \frac{\theta}{\sin\theta}\text{ lies between }\frac{\sin\theta}{\sin\theta}\text{ and }\frac{\tan\theta}{\sin\theta},$$

$$\textit{i.e.}\quad\text{between}\quad 1\quad\text{and }\frac{1}{\cos\theta}.$$

But, when θ is infinitely small, $\cos\theta = \cos 0 = 1$, hence, in this case,

$$\frac{\theta}{\sin\theta}\text{ lies between 1 and 1}$$

and therefore
$$\frac{\theta}{\sin\theta} = 1.$$

Again, $$\frac{\theta}{\tan\theta}\text{ lies between }\frac{\sin\theta}{\tan\theta}\text{ and }\frac{\tan\theta}{\tan\theta},$$

i.e. between $\cos\theta$ and 1.

Hence, when θ is infinitely small,

$$\frac{\theta}{\tan\theta} = 1.$$

So, too, the reciprocals of these ratios are also equal to unity, *i.e.*

$$\frac{\sin\theta}{\theta}=1 \quad \text{and} \quad \frac{\tan\theta}{\theta}=1 \quad \ldots\ldots\ldots\ldots(138)$$

COR. Hence, when θ is small, $\sin\theta$ and $\tan\theta$ are approximately equal to θ. In other words, *the sine and tangent of a small angle are both approximately equal to its circular measure.*

NOTE.—The fractions $\frac{\theta}{\sin\theta}$ and $\frac{\theta}{\tan\theta}$ assume the form $\frac{0}{0}$ when $\theta=0$. They are thus *vanishing fractions*, whose values are found above by regarding them as limits.

ILLUSTRATIVE EXERCISE.

Point out the fallacy of the following proof:—

When $\theta=0$, $\sin\theta=0$; $\therefore$ $\theta=\sin\theta$ and $\frac{\sin\theta}{\theta}=1$.

Ex. 1. To find the limits of $\frac{\sin A^\circ}{A}$ and $\frac{\tan A^\circ}{A}$, when A, the measure of an angle in *degrees*, is made infinitely small.

If θ be the circular measure of A°, then

$$\theta=\frac{\pi}{180}A.$$

Also the sine of an angle is independent of the unit of angular measurement;

$$\therefore \quad \frac{\sin A}{A}=\frac{\pi}{180}\frac{\sin\theta}{\theta}=\frac{\pi}{180}\times 1, \quad \text{or} \quad \frac{\pi}{180} \text{ is the limit.}$$

Similarly,
$$\frac{\tan A}{A}=\frac{\pi}{180}.$$

Ex. 2. To prove that, if n be the number of seconds in a very small angle,
$$\sin n''=\tan n''=\frac{n}{206{,}265}, \text{ approximately.}$$

Since n'' is a small angle, its sine and tangent are both approximately equal to its circular measure

$$=n\times\frac{\pi}{180\times 60\times 60}.$$

Taking $\pi=3{\cdot}14159\ldots$, we find

$$\frac{180\times 60\times 60}{\pi}=206{,}265,$$

approximately. This proves the result.

Ex. 3. Find, approximately, the height of a tower which subtends an angle of 13′ at the eye of an observer 5 miles distant (taking $\pi = \frac{22}{7}$).

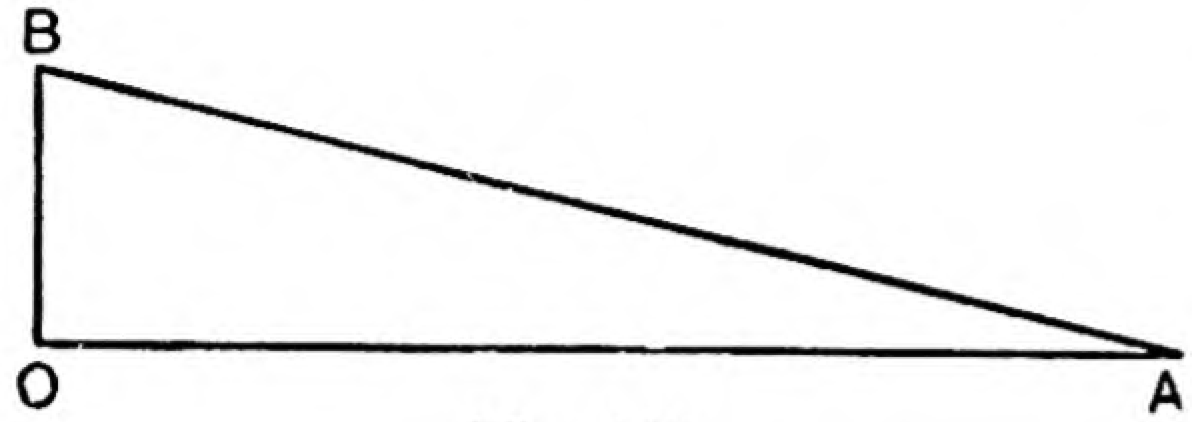


Fig. 143.

If **OB** represents the tower, **A** the observer, we have

$$\mathbf{OB} = \mathbf{AO} \tan \mathbf{OAB}$$
$$= 5 \text{ miles} \times \tan 13'.$$

Now $\tan 13' = \text{circ. measure of } 13' = \dfrac{13\times\pi}{180\times 60} = \dfrac{13\times 22}{180\times 60\times 7}$, approximately;

$\therefore$ height of tower $= \dfrac{5\times 5280 \text{ ft.}\times 13\times 22}{180\times 60\times 7} = \frac{6292}{63}$ ft. $= 100$ ft., nearly.

Ex. 4. Taking $\pi = \frac{22}{7}$, find the angle which the bull's-eye of a target, 6 in. in diameter, subtends at the eye of an observer 200 yd. off in front of its centre.

If 2θ be the required angle, we have (from a figure)

$$\tan\theta = \frac{\text{radius of bull's-eye}}{\text{distance of target}} = \frac{3 \text{ in.}}{200 \text{ yd.}} = \frac{\frac{1}{4}\text{ ft.}}{600 \text{ ft.}} = \tfrac{1}{2400};$$

$$\therefore \quad \theta = \tfrac{1}{2400} \text{ radian (approximately)};$$

$$\therefore \quad 2\theta = \frac{2\times 180^\circ}{\pi\times 2400} = \frac{3}{20\pi} \text{ degrees} = \frac{9}{\pi} \text{ min.} = \tfrac{63}{22} \text{ min.};$$

$$\therefore \quad \text{required angle} = 2\tfrac{19}{22}' = 2'\,52'' \text{ to the nearest second.}$$

251. To prove that $\cos\theta$ lies between $1-\frac{1}{2}\theta^2$ and 1.

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}.$$

Also $\sin\dfrac{\theta}{2} < \dfrac{\theta}{2}$; $\quad\therefore\quad 1-2\sin^2\dfrac{\theta}{2} > 1-2\left(\dfrac{\theta}{2}\right)^2$;

$$\therefore \quad \cos\theta > 1 - \frac{\theta^2}{2} \quad\text{..................}(139)$$

Also $\cos\theta < 1$. Therefore $\cos\theta$ lies between $1-\frac{1}{2}\theta^2$ and 1.

When θ is small, the value of $\cos\theta$ approximates much more closely to $1-\frac{1}{2}\theta^2$ than to 1. In fact, when θ is indefinitely diminished,

the limit of $\quad \dfrac{1-\cos\theta}{\theta^2} = \dfrac{2\sin^2\frac{1}{2}\theta}{\theta^2} = \frac{1}{2}\left(\dfrac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right)^2 = \frac{1}{2}$;

and therefore $1-\cos\theta$ approximates to $\frac{1}{2}\theta^2$, and $\cos\theta$ to $1-\frac{1}{2}\theta^2$.

Ex. 1. If the dip of the horizon at sea be $7'$, find the distance of the offing and the height of the observer, taking the earth as a sphere of radius 4,000 miles.

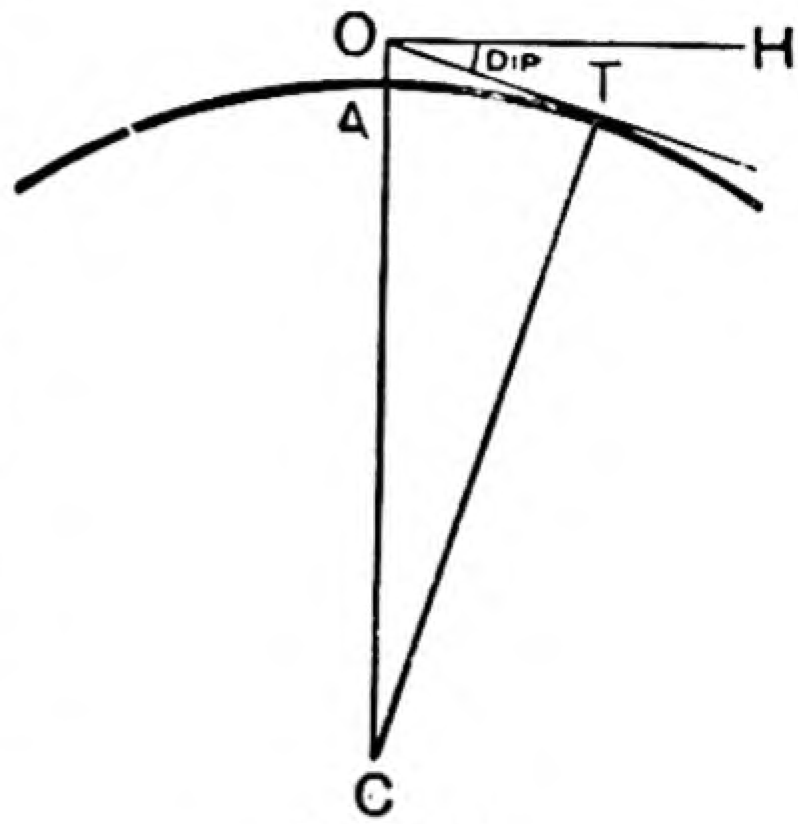


Fig. 144.

Let **O** be the observer, and let **OT** be the tangent from **O** to the spherical surface of the sea. Then **T** lies on the boundary of that part of the surface visible from **O**. This boundary is called the **offing**, and if **OH** is horizontal at **O** (*i.e.* perpendicular to the direction of gravity at **O**), *the angle of depression* **HOT** is called the **dip of the horizon.**

Let **C** be the centre of the earth. Then **OH** is perpendicular to **OC**; hence

$$\angle \mathbf{TCO} = \angle \mathbf{TOH} = 7' \text{ (by data).}$$

Let θ be the circular measure of $7'$.

Then **OT** the distance of the offing

$$= 4000 \text{ miles} \times \tan 7' = 4000 \text{ miles} \times \theta \text{ (approximately).}$$

Now $$\theta = \frac{7\times\pi}{180\times 60} = \frac{22}{180\times 60} \text{ roughly.}$$

Hence $$\mathbf{OT} = \frac{4000\times 22}{180\times 60} \text{ miles} = 8{\cdot}15 \text{ miles roughly.}$$

Again, if **A** is on the sea-level vertically below **O**, $\mathbf{CO} = \mathbf{CT} \sec 7'$, and $\mathbf{AO} = \mathbf{CO} - \mathbf{CT} = \mathbf{CT}(\sec 7' - 1)$.

Now $$\sec 7' - 1 = \frac{1-\cos 7'}{\cos 7'} = \frac{1-(1-\frac{1}{2}\theta^2)}{1-\frac{1}{2}\theta^2} = \frac{\frac{1}{2}\theta^2}{1-\frac{1}{2}\theta^2}$$

$$= \frac{\frac{1}{2}\theta^2}{1} \text{ (approximately)} = \tfrac{1}{2}\left(\frac{22}{180\times 60}\right)^2;$$

hence $$\mathbf{OA} = 2000 \text{ miles} \times \left(\frac{22}{180\times 60}\right)^2 = \frac{2000\times 5280\times 11\times 11}{90\times 90\times 60\times 60} \text{ ft.};$$

$\therefore$ height of observer above sea-level $= 43{\cdot}8$ ft. approximately.

Ex. 2. Solve, approximately, the triangle in which $a = 5$ ft., $A = 10'$, $B = 150°$.

Here $$b = a\frac{\sin B}{\sin A} = 5\,\frac{\sin 150°}{\sin 10'}$$

$$= \frac{5\times\frac{1}{2}}{\text{circular measure of } 10'} \text{ (approximately)}$$

$$= \frac{5}{2}\times\frac{180\times 60}{10\times\pi} = \frac{90\times 30\times 7}{22} \text{ (taking } \pi = \tfrac{22}{7}\text{)}$$

$$= 859{\cdot}1 \text{ ft. nearly,}$$

$$c = b\cos A + a\cos B = 859{\cdot}1 \cos 10' - 5\times\tfrac{1}{2}\sqrt{3}.$$

For a first approximation we may take $\cos 10' = 1$.

[For, if θ be the circular measure of $10'$, we have, roughly,

$$\theta = \frac{10\times 22}{180\times 60\times 7} = \frac{11}{3780},$$

and $\cos\theta$ differs from unity by less than $\frac{\theta^2}{2}$, or $\frac{1}{2}\left(\frac{11}{3780}\right)^2$, which may be neglected.]

Hence $$c = 859{\cdot}1 - \tfrac{5}{2}\sqrt{3} = 859{\cdot}1 - 4{\cdot}3$$
$$= 854{\cdot}8 \text{ ft. nearly.}$$

252. To prove that $\sin\theta$ lies between $\theta - \frac{1}{4}\theta^3$ and θ.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}$$

$$= 2\tan\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right).$$

Now $2\tan\frac{\theta}{2} > 2\frac{\theta}{2}$, or θ; also $\sin\frac{\theta}{2} < \frac{\theta}{2}$;

$$\therefore\quad 1-\sin^2\frac{\theta}{2} > 1-\left(\frac{\theta}{2}\right)^2,\quad i.e. > 1-\frac{\theta^2}{4};$$

$$\therefore\quad \sin\theta > \theta\left(1-\frac{\theta^2}{4}\right),\quad i.e. > \theta-\frac{\theta^3}{4} \quad\text{.........(140)}$$

But we have proved that $\sin\theta<\theta$. Therefore $\sin\theta$ lies between the limits θ and $\theta-\frac{1}{4}\theta^3$.

***253. To prove that $\sin\theta > \theta - \frac{1}{6}\theta^3$.**

Now $$\sin\theta = 3\sin\tfrac{1}{3}\theta - 4\sin^3\tfrac{1}{3}\theta.$$

But $$\sin\tfrac{1}{3}\theta = 3\sin\tfrac{1}{9}\theta - 4\sin^3\tfrac{1}{9}\theta,$$

$\therefore$ $$\sin\theta = 3^2\sin\tfrac{1}{9}\theta - 4\{\sin^3\tfrac{1}{3}\theta + 3\sin^3\tfrac{1}{9}\theta\}.$$

Substituting again $$\sin\tfrac{1}{9}\theta = 3\sin\tfrac{1}{27}\theta - 4\sin^3\tfrac{1}{27}\theta$$

$$\sin\theta = 3^3\sin(\theta/3^3) - 4\{\sin^3(\theta/3) + 3\sin^3(\theta/3^2) + 3^2\sin^3(\theta/3^3)\}.$$

Proceeding in this manner, we obtain

$$\sin\theta = 3^n\sin(\theta/3^n) - 4\{\sin^3(\theta/3) + 3\sin^3(\theta/3^2) + 3^2\sin^3(\theta/3^3) + \dots + 3^{n-1}\sin^3(\theta/3^n)\}.$$

But $$\sin^3\tfrac{1}{3}\theta < (\tfrac{1}{3}\theta)^3,\ \sin^3(\theta/3^2) < (\theta/3^2)^3,\ \dots,\ \sin^3(\theta/3^n) < (\theta/3^n)^3.$$

Hence $$\sin\theta > 3^n\sin(\theta/3^n) - 4\theta^3\left\{\frac{1}{3^3}+\frac{1}{3^5}+\frac{1}{3^7}+\dots+\frac{1}{3^{2n+1}}\right\}.$$

The expression in brackets on the right is a geometrical progression whose sum is $\dfrac{1/3^3 - 1/3^{2n+3}}{1-1/3^2}$, and is therefore less than $\dfrac{1/3^3}{1-1/3^2}$ for all values of n. Hence for all values of n,

$$\sin\theta > 3^n.\sin(\theta/3^n) - 4\theta^3.\frac{1/27}{1-1/9}$$

$$i.e.\ \sin\theta > \theta.\frac{\sin(\theta/3^n)}{(\theta/3^n)} - \frac{\theta^3}{6}.$$

Let n tend to infinity. Then $\theta/3^n$ tends to zero and $\dfrac{\sin(\theta/3^n)}{\theta/3^n}$ becomes unity. Hence in the limit the proposed inequality is satisfied, *i.e.*

$$\sin\theta > \theta - \frac{\theta^3}{6} \quad \text{.............................(141)}$$

***254. To prove that $\cos\theta < 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$.**

The proof is similar to that of § 251.

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$< 1 - 2\left\{\frac{\theta}{2} - \frac{1}{6}\left(\frac{\theta}{2}\right)^3\right\}^2, \text{ from the last article,}$$

i.e. $$< 1 - 2\left\{\frac{\theta^2}{4} - \frac{1}{6}\frac{\theta^4}{8} + \frac{1}{36}\frac{\theta^6}{64}\right\},$$

i.e. $$< 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{1152},$$

and $$\therefore\ \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} \quad \text{.....................................(142)}$$

EXAMPLES XXIII.

1. Find the area of a circle whose radius is 20 ft.; also a sector of the same circle whose angle is 9°. (Take $\pi = \frac{22}{7}$.)

2. Find the area of an annulus bounded by two circles whose radii are 2·87 ft. and 1·13 ft., respectively.

3. A square has 6 in. side. Find the area included between its incircle and its circumcircle.

4. Find the area of a circle whose circumference is 132 centimetres.

5. The sector of a circle with radii 9 in. has area 162 sq. in. Find its angle.

6. An equilateral triangle is inscribed in a circle of radius 5 in. Find the area of each of the minor segments thereby produced.

7. Two equal circles are drawn with the centre of the one on the circumference of the other. Find the ratio of the area of the part of one circle which falls on the other to that of the part which falls outside the other.

8. Six equal circles of radius r are placed so that each touches two others, their centres all being on the circumference of another circle. Find the area which they enclose.

9. Prove that the area of a regular polygon of n sides, inscribed in a circle, is $\frac{1}{2}nr^2 \sin\left(\frac{360°}{n}\right)$. If $n = 540$, find, by means of a table of logs, the ratio of the area of the polygon to the area of the circle, taking $\pi = 3{\cdot}141$, and given

$$L \sin 40' = 8{\cdot}06578.$$

10. The area of a regular polygon of n sides of given perimeter p is $\frac{p^2}{4n}\cot\frac{\pi}{n}$. Show that, the greater the number of sides, the greater the area of the polygon. Find its greatest and least values.

11. Show that the perimeter of a triangle is to the perimeter of the inscribed circle as the area of the triangle is to the area of the circle.

12. Show that the area of a triangle **ABC** is to the area of the inscribed circle as $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ is to π.

13. Three equal circles, radius 1 ft., touch each other externally. Find the area of the curvilinear figure formed between the circles.

14. Two circles of radii r and $3r$ touch each other externally, and a common tangent is drawn. Find the area of the curvilinear triangle included by the tangent and the two arcs of the circles.

15. A length of 200 yd. of cloth, whose thickness is $\frac{1}{30}$ in., is rolled up into a cylinder. Find the diameter of the cylinder.

16. Two miles of paper are rolled up into a cylinder. The thickness of the paper is $\frac{1}{200}$ in. Find the diameter of the cylinder.

17. If A be the area of the inscribed circle of a triangle, A_1, A_2, A_3, the areas of the three escribed circles, then will

$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}.$$

18. If x is the circular measure of a very small angle, show that

$$\sin(A+x) = \sin A + x\cos A, \text{ nearly,}$$

and obtain a similar expression for $\cos(A+x)$.

19. Hence, show that

$$\frac{\sin(A+x)-\sin A}{\cos A-\cos(A+x)} = \cot A \text{ nearly.}$$

20. If circles be inscribed in and described about two regular polygons of the same perimeter, the second of which has twice as many sides as the first, then (i) the radius of the incircle of the second is an arithmetic mean between the radii of the incircle and circumcircle of the first, and (ii) the radius of the circumcircle of the second is a geometric mean between the radii of the incircle of the second and the circumcircle of the first.

21. Apply the results of Question 20 to calculate in order the radii of the in- and circumcircles of the polygons of 8, 16, 32, 128 sides, which have the same perimeter as a square, each of whose sides is 1 unit long.

22. Show how the method of the last example can be made to yield a value for π, and by its means calculate π from a polygon of 128 sides.

23. Show how to calculate the value of the sine of a small angle, and find the value of $\sin 20''$ to 6 decimal places.

24. A church tower at a distance of 2 miles subtends an angle of $1° 5' 6''$. Find, approximately, its height.

25. If the Earth's radius (3,960 miles) subtend at the centre of the Sun an angle of $8{\cdot}57116''$, determine the Sun's distance from the Earth.

26. Given the Sun's apparent diameter to be $31\frac{1}{2}'$, determine his actual diameter in miles, assuming his distance to be 96,000,000 miles.

27. The Earth's radius (3,960 miles) subtends at the Moon an angle of $57' 1{\cdot}8''$. Find the Moon's distance from the Earth.

28. The radius of the Earth's orbit, supposed circular, being 96,000,000 miles, find the distance of a fixed star at which the diameter of the Earth's orbit subtends an angle of $\cdot 8''$.

29. The Sun's distance from the Earth being 24,000 times the Earth's radius, find in seconds the Earth's apparent diameter as seen from the Sun.

30. Taking the Sun's apparent diameter as $31\frac{1}{2}'$, and his distance from the Earth as 96,000,000 miles, show that, if he were concentric with the Earth, his body would extend in all directions 200,000 miles beyond the Moon, whose distance from the Earth is 240,000 miles.

31. Taking the Sun's diameter as 880,000 miles, and the Earth's as 8,000, compare the apparent magnitudes of the Sun and Earth as seen from each other.

32. The apparent angular diameter of the Sun is $30'$. A planet is seen to cross the disc in a straight line, at a distance from the centre equal to $\frac{3}{5}$ of the radius. Prove that the angle subtended at the Earth by the part of the planet's path projected on the Sun is $\dfrac{\pi}{450}$.

33. Explain the method of calculating the numerical value of the cosine of a very small angle.

34. At what distance would an object 8 in. high be hid by the curvature of the Earth over the surface of still water? (Radius of Earth = 3,960 miles.)

35. Find the dip of the Earth's horizon if its distance be 3 miles.

36. What is the furthest distance at which the Peak of Teneriffe, $2\frac{1}{2}$ miles above sea-level, can be seen at sea by an eye on the water-surface?

37. If a mountain 6,600 ft. high could be seen at a distance of 100 miles, what would be the Earth's radius?

38. At what distance could the tops of two ships' masts first be sighted from each other, if their height is 80 ft.?

39. A ship's hull stands 30 ft. out of the water, and her mast is 90 ft. high. At what distance will her hull be lost to sight, and how much further can she sail before she altogether disappears, to an eye on the water-surface?

40. From the top of a cliff the angle of depression of the horizon is $10'$. Show that the height of the cliff is $\frac{1}{2}\frac{\pi^2 r}{1080^2}$ where r is the Earth's radius.

41. In the last question, taking $\pi = 3{\cdot}1416$, $r = 3990$ miles, find the height in yards, using a table of logs.

42. From the mast of a ship, the top of a light-house, known to be 500 ft. above sea-level, is just visible at a depression of $9'\,27''$. How far is the ship from the lighthouse?

43. Assuming $\sin\theta$ to be equal to $\theta - \frac{1}{6}\theta^3$, find the value of $\tan\theta$ in powers of θ, neglecting θ^5 and higher powers.

44. Prove that $$\theta < \tan\theta < \theta + \theta^3/4$$ if θ is between 0 and $\frac{1}{2}\pi$.

ANSWERS.

Examples I. (Page 5.)

1. $\frac{1}{16}$; $392\frac{1}{2}°$. 2. $2° 48' 45''$; $402\frac{1}{2}°$. 3. $43\frac{7}{11}'$ past **8.**

4. $49° 15' 36''$; $32° 58' 1{\cdot}2''$; $85° 38' 54{\cdot}6''$.

5. ·4695; ·703571: ·0555525. 6. $81° 41' 42''$.

7. 45°, 45°, 90°. **8.** 90°, 70°, 20°. **9.** 30° 60° 90°.

11. 120°, 144°, 156°. **13.** 1 : 3. **14.** 54°, 81°, 108°, 135°, 162°.

15. 9 in. **16.** $51{\cdot}84''$. **17.** $14\frac{4}{11}'$, $20\frac{4}{11}'$, past 3.

18. 2037 yd. **19.** $11\frac{1}{4}°$. **20.** 756°.

21. 36°, 324°. **22.** $5400n/(60n+1)$. **23.** 160°, 140°.

24. 20. **25.** $\frac{14}{27}$. **26.** 135°, 90°.

27. 8 or 24 right angles, according as it rolls inside or outside.

28. 144°. **29.** $55^g\, 25'$; $25^g\, 78'\, 2''$; $21^g\, 5'\, 65''$.

30. $28° 21' 35{\cdot}6''$; $7° 29' 20{\cdot}4''$, $12° 54' 59{\cdot}2''$. **31.** Equal.

32. 4th, 3rd, 1st, 1st. **33.** 45°.

Examples II. (Page 14.)

1. 3438. **4.** $38° 11' 50''$. **5.** 1·5708. **6.** 1·9673.

7. 301 ft. **8.** ·6501. **9.** 69·36 ft. **10.** $25' 47''$.

11. 24010. **12.** ·004 in. **13.** 859 ft. **14.** 33 ft.; 207 ft. 4 in.

15. 3·1416 : 1. **16.** 56. **17.** $3° 11'$.

18. 12·566 in.; ·785 in.; 2 in.; $16\frac{4}{11}'$ past 12.

19. ·1068; $40° 6' 25''$. **20.** 2·443; 140°. **21.** A right angle.

22. ·6981, 1·0472, 1·3963; 40°, 60°, 80°. **23.** 21·22.

24. 1·0966; $62° 49' 55''$. **25.** $\frac{1}{12}\pi$, $\frac{1}{3}\pi$, $\frac{7}{12}\pi$. **26.** $\pi - 2$.

27. $\frac{355}{113}$. **28.** ·6366. **29.** $\frac{1}{2}\pi$, $\frac{2}{3}\pi$. **31.** $1° 1' 4''$; $\pi/(180-\pi)$.

32. 80°, 40°, 60°. **33.** 30°, 60°, 90°. **34.** 3·1416.

35. ·3708; $21° 14' 43''$. **36.** 2 ml. 1600 yd. **37.** 135°, 150^g, $\frac{3}{4}\pi$.

38. 4 ft. $8\frac{1}{2}$ in. **39.** 2,774,724 ml. **40.** 28·56.

41. 55° 24′ 30″. **42.** 2. **43.** $(n-2)\pi/n$.

45. $\frac{1}{360}\pi$, $\frac{1}{21600}\pi$. **46.** 1 ft. 6½ in. **47.** ·2529 ml., ·0015 ml.

48. 31·416 ml. per hr. **49.** 57′ 18″. **50.** 9 ft. 6½ in.

51. 3·15. **52.** 69 ml. **53.** 4 ft. 8 in. **54.** 19.

Examples III. (Page 27.)

1. ·75; 2·4. **2.** $C = 70°$, $a = 29·12$, $b = 85·13$.

3. $C = 90°$, $b = 1143$, $c = 1147$. **4.** $A = 30°$, $B = 60°$, $b = 17·32$.

5. $B = 90°$, $a = 1·04$, $c = 5·9$. **6.** $B = 25°$, $a = 44·14$, $b = 18·65$.

7. $A = 45°$, $C = 90°$, $c = 42·43$. **8.** $C = 90°$, $b = 1428·1$, $c = 1743·4$.

9. $C = 75°$, $a = 18·75$, $b = 72·47$. **10.** $A = 60°$, $a = 38·1$, $c = 44$.

11. $A = 60°$, $B = 30°$, $c = 1000$. **12.** $B = 10°$, $C = 80°$, $a = 10000$.

13. $A = 90°$, $a = 1305$, $c = 1000$. **14.** $C = 65°$, $b=227·17$, $c=205·87$.

15. $A = 55°$, $a = 10$, $c = 12·2$. **16.** $B = 45°$, $b = c = 7·78$.

17. $C = 40°$, $a = 21·45$, $b = 28$. **18.** $C = 90°$, $a = 2·18$, $b = 24·9$.

19. $B = 85°$, $C = 5°$, $b = 1245$. **20.** $A = 70°$, $B = 20°$, $a = 469·85$.

21. $B = 20°$, $C = 70°$, $c = 2349$. **22.** $B = 60°$, $a = 10$, $b = 17·32$.

23. 60°. **24.** ·1232, 1, ·5, 1. **25.** 102·6 ft. **26.** 11·78 ft.

27. 37·32 ft. **28.** 1922 ft. **29.** 329·7 ft. **30.** 29·12 ft.

31. 20 ft. **32.** 45 ft. **33.** 67·2 ft. **34.** 5°.

35. 90 ft. **36.** 274 ft. **37.** 201½ yd. **38.** ·38, ·92, ·41.

39. ·87, ·50, 1·73. **40.** ·61, ·79, ·77.

Examples IV. (Page 45.)

7. Negative; positive.

8. Positive in 1st and 4th quadrants; negative in 2nd and 3rd.

12. ·992. **15.** sin, cosec, tan, cot.

16. (i) sin, cos; (ii) cosec, sec. **17.** No. **18.** No.

19. Yes; the fraction is improper. **20.** Only when $a = b$.

Examples V. (Page 61.)

5. 14° 30′, 165° 30′; 30°, 150°; 48° 30′, 131° 30′.

7. From 1 to -1 and back. **9.** $\frac{3\pi}{4}, \frac{7\pi}{4}$; or 135°, 315°.

Examples VI. (Page 69.)

7. 1. **8.** 1. **9.** 2. **10.** $\frac{1}{4}$.

11. $2-\sqrt{\frac{2}{3}}$. **12.** 1. **13.** $-(2+\sqrt{3})$. **14.** 1.

15. 0. **16.** 1. **17.** $2\frac{1}{2}$. **18.** ∞.

19. $\frac{1}{2}$. **20.** $2+\frac{5}{6}\sqrt{3}$. **21.** 1. **22.** $4\frac{1}{3}$.

23. $\frac{1}{2}\sqrt{2}$. **30.** $\frac{1}{4}\sqrt{2}$. **31.** $\sqrt{\frac{2}{3}}$. **32.** $\frac{1}{4}$.

33. $2\frac{2}{3}$. **34.** -1. **36.** $\frac{40}{3}\sqrt{3}$, $\frac{20}{3}\sqrt{3}$, 20 ft. **37.** $6\sqrt{3}$ ft.

38. 60°. **39.** 12·68 ft. **40.** $50\sqrt{2}$ yd. **41.** $6\sqrt{3}$ ft.

42. $\frac{5}{2}\sqrt{3}$ ml.; $2\frac{1}{2}$ ml. **43.** $\frac{5}{4}\sqrt{6}$ ml. **44.** 30 ft.

45. $50\sqrt{3}$ yd.

46. Left edge, ·06; base, ·08; right edge, ·0598; top, ·07986.

47. $50\sqrt{3}$ or $100\sqrt{3}$ yd.

48. $1320\sqrt{6}$ ft. **49.** 764 ft. **50.** 162 ft. **51.** $50\sqrt{3}$ ft.

Examples VII. (Page 85.)

2. $\frac{5}{13}$, $\frac{12}{5}$. **3.** $\frac{1}{4}\sqrt{7}$, $\frac{1}{3}\sqrt{7}$, $\frac{3}{7}\sqrt{7}$. **4.** $\pm\frac{4}{5}$, $\mp\frac{3}{5}$, $-\frac{3}{4}$.

5. $\sin A = \pm\frac{1}{2}\sqrt{3}$, $\cos A = \frac{1}{2}$, $\tan A = \pm\sqrt{3}$, $\cot A = \pm\frac{1}{3}\sqrt{3}$,
$\operatorname{cosec} A = \pm\frac{2}{3}\sqrt{3}$.

5*a*. 3rd or 4th, $\cos A = -\frac{4}{5}$, $\cot A = \frac{4}{3}$; $\cos A = \frac{4}{5}$, $\cot A = -\frac{4}{3}$.

5*b*. 2nd or 3rd. $\operatorname{Cosec} A = \sqrt{2}$, $\tan A = -1$;
$\operatorname{cosec} A = -\sqrt{2}$, $\tan A = 1$.

5*c*. 1st or 3rd, $\cos A = b/\sqrt{(a^2+b^2)}$, $\operatorname{cosec} A = \sqrt{(a^2+b^2)}/a$;
$\cos A = -b/\sqrt{(a^2+b^2)}$, $\operatorname{cosec} A = -\sqrt{(a^2+b^2)}/a$.

5*d*. 3rd or 4th, $\sec A = -l/\sqrt{(l^2-m^2)}$, $\tan A = m/\sqrt{(l^2-m^2)}$;
$\sec A = l/\sqrt{(l^2-m^2)}$, $\tan A = -m/\sqrt{(l^2-m^2)}$.

5*e*. (i) +, in the first and second quadrants; −, in the third and fourth. (ii) +, in the first and third quadrants; −, in the second and fourth. (iii) +, in the first and second quadrants; −, in the third and fourth.

37. $\sec\theta$. **38.** $\dfrac{1-\sin^2\theta\cos^2\theta}{2+\sin^2\theta\cos^2\theta}$. **39.** $\sec\theta \operatorname{cosec}\theta$.

40. 1. **41.** 60 ft., $22\frac{1}{2}$ ft. **42.** 24 ft., 34 ft.

Examples VIII. (Page 101.)

7. 2·5, ·928, ·371. **8.** $-\frac{1}{2}\sqrt{3}$, -1, $2-\sqrt{3}$. **10.** $-\frac{2}{3}\sqrt{3}$, $\sqrt{3}$, -1.

11. $-\frac{1}{2}$, $-\sqrt{2}$, $-\frac{1}{3}\sqrt{3}$. **12.** $\frac{1}{2}\sqrt{2}$, $\frac{1}{3}\sqrt{3}$, $\frac{1}{2}$.

13. $-\frac{1}{2}\sqrt{3}$, $-\sqrt{2}$, $-\sqrt{3}$. **14.** $1+\frac{1}{2}\sqrt{2}$, $\sqrt{3}$, $\frac{2}{3}\sqrt{3}$.

15. $-\sqrt{2}, 1+\frac{1}{2}\sqrt{3}, 0.$ **16.** $2n\pi+\frac{1}{4}\pi$ and $(2n+1)\pi+\frac{1}{4}\pi.$

17. $323° 7'$; $143° 7'$.

18. From 1 to $-\infty$; then changing sign and descending from ∞ to -1.

19. From $\sqrt{3}$ up to 2, and down to $-\sqrt{3}$.

Examples IX. (Page 114.)

3. $\sqrt{(x^2-1)}$. **4.** No; the first is an angle, the second not.

6. $x/\sqrt{(1-2x^2)}, a, \sqrt{\{(x^2+1)/(x^2+2)\}}$. **7.** $\frac{5}{9}$. **8.** $n\pi+(-1)^n\frac{1}{6}\pi$.

10. An angle in the 3rd quadrant whose tangent is $\frac{3}{4}$. **11.** $A\pm B = n\pi$.

12. $69° 17' 42{\cdot}7''$. **13.** (a) $120°$; (b) $135°$. **14.** $153° 26', 333° 26'$.

15. $n\pi+(-1)^n \alpha$, α being any angle whose sine is $\frac{1}{3}$.

16. $(a^2-1)/(a^2+1)$. **17.** $\frac{1}{2}$ or 2. **21.** $n\pi$ or $2n\pi\pm\frac{1}{3}\pi$.

22. $(2n+1)\frac{1}{4}\pi$. **23.** $2n\pi-\frac{1}{2}\pi$. **24.** $(2n+1)\frac{1}{4}\pi$.

25. $2n\pi-\frac{1}{2}\pi$ or $\frac{1}{5}(2n\pi+\frac{1}{2}\pi)$. **26.** $n\pi/\{4-(-1)^n3\}$.

27. $\frac{1}{5}n\pi+(-1)^n\frac{2}{45}\pi$. **28.** $\phi = n\pi$, $\theta = n\pi+\frac{1}{2}\pi$.

29. $A = (2m+n)\frac{1}{2}\pi+\frac{1}{3}\pi$, $B = (2m-n)\frac{1}{2}\pi+\frac{1}{6}\pi$.

31. $\sin(\theta+\frac{1}{4}\pi) = \cos(\theta-\frac{1}{4}\pi)$.

Examples X. (Page 121.)

1. (a) $2n\pi\pm\frac{1}{4}\pi$; (b) $\frac{1}{2}n\pi+\frac{1}{8}\pi$. **2.** $2n\pi$. **3.** $2n\pi\pm\frac{1}{6}\pi$.

4. $n\pi\pm\frac{1}{3}\pi$ or $n\pi\pm\frac{1}{4}\pi$. **5.** $\frac{1}{2}n\pi\pm\frac{1}{12}\pi$. **6.** $n\pi\pm\frac{1}{6}\pi$ or $n\pi\pm\frac{1}{4}\pi$.

7. $n\pi\pm\frac{1}{6}\pi$ or $n\pi\pm\frac{1}{3}\pi$. **8.** $n\pi\pm\frac{1}{3}\pi$ or $\frac{1}{2}n\pi+\frac{1}{4}\pi$. **9.** $n\pi+(-1)^n\frac{1}{6}\pi$.

10. $2n\pi+\frac{1}{4}\pi\pm\frac{1}{4}\pi$. **11.** $(2n+1)\frac{1}{4}\pi$. **12.** $2n\pi\pm\frac{1}{3}\pi$.

13. $n\pi\pm\frac{1}{6}\pi$ or $\frac{1}{2}n\pi+\frac{1}{4}\pi$. **14.** $2n\pi+\frac{1}{4}\pi\pm\frac{1}{4}\pi$.

15. $2n\pi\pm\frac{1}{3}\pi$ or $\cos^{-1}(-\frac{3}{4})$. **16.** $n\pi+(-1)^n\frac{1}{6}\pi$. **17.** $n\pi+(-1)^n\frac{1}{3}\pi$.

18 & 19. $n\pi+(-1)^n\frac{1}{6}\pi$. **20.** $\frac{1}{2}n\pi+\frac{1}{8}\pi$. **21.** $\frac{1}{2}n\pi+\alpha$.

22. $2n\pi\pm\frac{1}{3}\pi$ or $(2n+1)\pi$. **23.** $n\pi\pm\frac{1}{3}\pi$. **24.** $2n\pi+\frac{1}{6}\pi$.

25. $\tan^{-1}\frac{1}{2}$. **26.** $n\pi+(-1)^n\frac{1}{6}\pi$. **27.** $n\pi+(-1)^n\frac{1}{6}\pi$.

28. $\sin x = \frac{3}{5}, \frac{1}{3}, -\frac{1}{3}, -\frac{3}{5}$; $\sin y = \frac{1}{3}, \frac{3}{5}, -\frac{3}{5}, -\frac{1}{3}$.

29. $a^2+b^2 = c^2+d^2$. **30.** $a^2b^2 = a^2+1$. **31.** $a^2b^2 = a^2+b^2$.

32. $(ab-c^2)^2+(bc-a^2)^2 = (ca-b^2)^2$. **33.** $a^2+b^2 = c^2+d^2$.

34. $a^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}}) = 1$. **35.** $a^2+2c = 1$.

36. $(a^2-b^2)^2(a^2x^2-b^2y^2)^2 = 2(a^2x^2+b^2y^2)^3$. **37.** $p^2+s^2 = q^2+r^2$.

38. $(pn'+nm')(p'n+n'm) = (pp'-mm')^2$.

39. $a(c^2-b^2)^2 = 4b(2c-ab)$. **40.** $(aa'-cc')^2 = (ab'-bc')(a'b-b'c)$.

Examples XI. (Page 139.)

7. $\dfrac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot A\cot B+\cot B\cot C+\cot C\cot A-1}$. **8.** $\dfrac{a+b+c-abc}{1-ab-bc-ca}$.

10. $\frac{33}{65}$. **11.** $\sin(A+B)=\frac{980}{2501}$, $\cos(A+B)=-\frac{2301}{2501}$,
$\sin(A-B)=\frac{100}{2501}$, $\cos(A-B)=\frac{2499}{2501}$.

12. 2. **14.** $\sqrt{(a^2+b^2)}$; $\theta=\tan^{-1}(b/a)$. **15.** $\frac{1}{2}n\pi+(-1)^n A$.

Examples XII. (Page 149.)

2. $n\pi$. **3.** 3, $\frac{9}{13}$. **6.** $\{\frac{1}{2}\pm\frac{1}{4}\sqrt{[2(n+1)]}\}^{\frac{1}{2}}$; $\frac{1}{2}\sqrt{(2-\sqrt{2})}$; $\frac{1}{2}\sqrt{(2+\sqrt{2})}$.

7. $\{-1\pm\sqrt{(n^2+1)}\}/n$; 30°, 120°, 210°, 300°.

12. $\sin\frac{1}{2}A=\frac{3}{58}\sqrt{58}, \frac{7}{58}\sqrt{58}, -\frac{3}{58}\sqrt{58}, -\frac{7}{58}\sqrt{58}$;
$\cos\frac{1}{2}A=\frac{7}{58}\sqrt{58}, -\frac{3}{58}\sqrt{58}, -\frac{7}{58}\sqrt{58}, \frac{3}{58}\sqrt{58}$;
$\tan\frac{1}{2}A=\frac{3}{7}, -\frac{7}{3}$.

14. Negative in each case.

Examples XIII. (Page 164).

3. $-\cot 6\theta$. **4.** $-\tan\theta$. **5.** $\tan 4\theta$. **6.** $\tan\frac{5}{2}A$. **7.** $\tan 4A$.

8. (i) $(2n+1)\pi\pm\frac{1}{2}\pi$; (ii) between $2n\pi$ and $2n\pi+\frac{1}{4}\pi$, $(2n+1)\pi-\frac{1}{4}\pi$ and $(2n+1)\pi$, $2n\pi+\frac{5}{4}\pi$ and $2n\pi+\frac{7}{4}\pi$.

Examples XIV. (Page 178.)

1. $\frac{1}{4}(\sqrt{5}-1)$, $\frac{1}{4}(\sqrt{5}+1)$. **2.** $2n\pi-\frac{1}{2}\pi$ or $\frac{1}{5}(2n\pi+\frac{1}{2}\pi)$.

3. $2n\pi$ or $\frac{1}{7}(2n+1)\pi$. **4.** $n\pi+\frac{1}{4}\pi$.

5. $(2n+\frac{1}{4})\pi\pm a$, where $\cos a=\pm(1, 3, \text{or } 5)\div 4\sqrt{2}$.

6. $n\pi$ or $n\pi+2a$, where $\tan a=2$.

7. $(2n+\frac{1}{4})\pi\pm a$, where $\tan a=\sqrt{7}$.

8. $n\pi+\frac{1}{4}\pi$, if a is not equal to $n\pi+\frac{1}{4}\pi$. **9.** $\frac{1}{3}n\pi$ or $\frac{2}{7}n\pi\pm\frac{1}{21}\pi$.

10. $\frac{1}{2}n\pi$ or $2n\pi\pm\frac{2}{3}\pi$. **11.** $\frac{1}{3}n\pi+\frac{1}{6}\pi$ or $n\pi\pm\frac{1}{3}\pi$.

12. $\frac{1}{3}n\pi$. **13.** $\frac{1}{3}n\pi+\frac{1}{6}\pi$ or $2n\pi\pm\frac{2}{3}\pi$. **14.** $\frac{1}{2}n\pi+\frac{1}{4}\pi$ or $2n\pi\pm\frac{2}{3}\pi$.

15. $(2n+1)\frac{1}{2}\pi$, $(2n+1)\pi$, $\frac{2}{5}n\pi$. **16.** $\frac{1}{4}n\pi$. **17.** $n\pi+\frac{1}{2}\pi$ or $\frac{2}{5}n\pi$.

18. $\frac{2}{3}n\pi$ or $\frac{1}{5}(2n+1)\pi$. **19.** $\frac{1}{2}\sin^{-1}\{4/(2n+1)\}$, $n>1$ or <-2.

20. $\frac{1}{2}n\pi+(-1)^n\frac{1}{12}\pi$. **21.** $(n+\frac{1}{2})\pi+2a$.

22. $\frac{1}{4}n\pi$. **23.** $n\pi+\frac{1}{2}\pi$ or $\frac{1}{5}n\pi+(-1)^n\frac{1}{30}\pi$.

24. $n\pi+(-1)^n\frac{1}{10}\pi$ or $n\pi-(-1)^n\frac{3}{10}\pi$. **25.** $n\pi+\frac{1}{2}\pi$ or $n\pi\pm\frac{1}{3}\pi$.

26. $2n\pi\pm\frac{1}{5}\pi$ or $2n\pi\pm\frac{3}{5}\pi$, if a is not equal to $n\pi$.

27. $2n\pi+\frac{1}{2}\pi$ or $2n\pi-\frac{1}{2}\pi-2a$, where $\tan a=\sqrt{2}$. **28.** $2n\pi$ or $2n\pi+\frac{1}{2}\pi$.

29. $(2n+1)\pi$ or $(2n+\frac{1}{2})\pi$. **30.** $n\pi+\frac{1}{4}\pi$ or $n\pi-\frac{1}{12}\pi$.
31. $2n\pi+a$ or $2n\pi-(\frac{1}{2}\pi+a)$. **32.** $2n\pi$ or $2n\pi+\frac{2}{3}\pi$.
33. $2n\pi+\frac{1}{2}\pi-a$ or $2n\pi-\frac{1}{2}\pi+a+2\beta$, where $\tan\beta=b/a$.
34. $2n\pi-a$. **35.** $2n\pi+a$ where $\tan a=2$. **36.** $n\pi$ or $2n\pi+\frac{1}{2}\pi$.
37. $n\pi-a$ or $2n\pi+a$. **38.** $\frac{1}{2}n\pi+\frac{1}{8}\pi$. **39.** $n\pi\pm\frac{1}{6}\pi$ or $\frac{1}{2}n\pi+\frac{1}{4}\pi$.
40. $(2n-\frac{1}{2})\pi\pm\frac{1}{4}\pi\pm a$. **41.** $2n\pi\pm\frac{1}{4}\pi\pm\beta$. **42.** $n\pi\pm a$.
43. $2n\pi$ or $2n\pi+\frac{1}{2}\pi$. **44.** $n\pi\pm\frac{1}{3}\pi$. **45.** $n\pi$ or $n\pi\pm\frac{1}{6}\pi$.
46. $n\pi+\frac{1}{4}\pi$ or $\frac{1}{2}n\pi+(-1)^n\frac{1}{12}\pi$. **47.** $(2n+1)\frac{1}{4}\pi$ or $(2n+1)\frac{1}{8}\pi$.
48. $n\pi+(-1)^n a$, where $a=\frac{1}{6}\pi$ or $\sin^{-1}(-\frac{1}{6})$.
49. $n\pi\pm\frac{1}{10}\pi$ or $n\pi\pm\frac{3}{10}\pi$. **50.** $n\pi+a$, where $a=\frac{1}{4}\pi$ or $\tan^{-1}3$.
51. $n\pi$ or $\frac{1}{2}n\pi+\frac{1}{4}\pi$. **52.** $n\pi+\frac{1}{4}\pi$ or $n\pi+\frac{1}{6}\pi$. **53.** $\frac{1}{3}n\pi+\frac{1}{12}\pi$.
54. $n\pi+a$, where $a=\frac{1}{4}\pi$ or $\tan^{-1}\frac{1}{2}$. **55.** $n\pi\pm\frac{1}{8}\pi$ or $n\pi$.
56. $n\pi+(-1)^n\frac{1}{10}\pi$ or $n\pi-(-1)^n\frac{3}{10}\pi$.
57. $n\pi+\frac{1}{2}\pi$, $2n\pi\pm\frac{1}{3}\pi$, or $2n\pi\pm\frac{1}{6}\pi$. **58.** $\frac{1}{2}n\pi-(-1)^n\frac{1}{12}\pi$.
59. $n\pi+\frac{1}{6}\pi$. **60.** $n\pi$. **61.** $2n\pi+\frac{1}{2}\pi$. **62.** $\frac{1}{2}n\pi$.
63. $\frac{1}{2}n\pi+\frac{1}{8}\pi$. **64.** $\frac{1}{2}n\pi\pm\frac{1}{6}\pi$. **65.** $2n\pi+a$. **66.** $\tan^{-1}\frac{19}{83}$.
73. $(1-6a^2+a^4)/(1+a^2)^2$. **114.** 6 or -2. **115.** $n\pi$ or $n\pi+\frac{1}{4}\pi$.
116. $\frac{5}{9}$. **117.** $\frac{1}{2}$. **118.** 0 or $\frac{1}{2}$. **119.** $\pm ab$.
120. $\sqrt{3}$. **121.** $\pm\sqrt{2}$. **122.** $\pm1\pm\sqrt{3}$.
123. $ab\div2\sqrt{(a^2\pm ab\sqrt{3}+b^2)}$. **124.** $\pm\sqrt{\{abc/(a+b+c)\}}$.
125. 0 or ±1. **126.** $\pm\sqrt{\frac{48}{7}}$.

Examples XV. (Page 193.)

5. 10. **6.** $\log(N\times a^n)=n+\log N$. **7.** $\bar{2}\cdot6197$.
8. $\bar{3}\cdot79588$. **9.** $\bar{1}\cdot90309$; $3\cdot60206$. **11.** 2·3219. **13.** ·5.
14. 4; ·25. **18.** 3, -1, 0, -4. **19.** -1, 9. **20.** 47.
23. 1, 3, 4. **24.** 0, 2, 4. **25.** $-\infty$ and 10.
26. 9·84949; 10·23856.

Examples XVI. (Page 214.)

1. 3·30103, $\bar{5}\cdot30103$, $\bar{3}\cdot59176$, 3·32193. **2.** ·72893.
3. 5·49987, 2·50004, 3·50087, $\bar{2}\cdot50101$. **4.** 3164·4, ·31706.
5. 9·62541, 9·96034, 10·33232, 10·39012, 10·38927, 9·66768.
6. 24° 4′, 54° 43′, 53° 37′, 11° 48′. **7.** 8·75125, 8·57746, 8·78149.
8. 8·63417, 11·25943, 11·23046, 8·57777.
9. 11·23122, 11·29101, 11·34614. **10.** 1·80718.

11. ·65533. **12.** 3° 17′, 2° 53′, 86° 1′. **13.** 87° 16′, 1° 10′, 2° 39′.
14. $\bar{1}$·69897. **15.** 1·1547, ·06247, 10·06247. **16.** 1·9307.
17. 2·92598. **18.** $\infty, -\infty$. **20.** 26° 14′. **21.** $x = 2, y = 3$.
22. 24° 14¼′. **23.** ·59176, $\bar{3}$·59176.
24. 3·73033, 6·73033, $\bar{3}$·73033. **25.** ·024408.
26. $\bar{1}$·79614, ·63864. **27.** ·45731, ·25038.
28. ·80937, 67° 31′. **29.** ·79031. **30.** $\bar{1}$·84952, ·61407.
31. ·41897. **32.** ·63630. **33.** −·87885, −·24233.
34. ·88288. **35.** 8·7542, ·38433. **36.** 19° 28′.
37. 9·46082, ·53754. **38.** 13° 21′. **39.** 102° 9′.
40. (*a*) 52° 1′ or 127° 59′, (*b*) 134° 46′, (*c*) 70° 52′ or 160° 52′, (*d*) 150° 38′.
41. 8·225 in. **42.** 1·0046.

Examples XVII. (Page 223.)

1.	$A = 50° 19'$,	$B = 39° 41'$,	$c = 1087·9$.
2.	$a = 1817$,	$c = 5254·0$,	$B = 69° 46'$.
3.	$a = 520$,	$b = 659$,	$B = 51° 44'$.
4.	$a = 654·2$,	$A = 64° 13'$,	$B = 25° 47'$.
5.	$c = 265·1$,	$A = 27° 45'$,	$B = 62° 15'$.
6.	$b = ·04767$,	$c = ·06223$,	$B = 50°$.
7.	$a = 35·76$,	$c = 27·31$,	$C = 37° 22'$.
8.	$a = 383·9$,	$b = 506·9$,	$C = 40° 46'$.
9.	$a = 3555·5$,	$c = 2354·7$,	$C = 33° 31'$.
10.	$b = 8418·5$,	$c = 8389·7$,	$C = 85° 16'$.
11.	$a = ·16782$,	$b = ·26109$,	$C = 50°$.
12.	$a = 2132·1$,	$B = 56° 29'$,	$C = 33° 31'$.

13. 25° 34′. **14.** 117·71 ft. **15.** 488·48 ft.
16. 1070·6 ft.; 1758·4 ft. **17.** 240·95 ft.; 29° 5′.
18. $A = 39° 20'$; $B = 50° 40'$; 14383·4 ft.
19. 16° 6′; 317800. **20.** 368·1 ft. **21.** 71·4 ft.
22. 18·199 ft. **23.** 66° 43′; 395·45 ft.

24.	$b = 54·929$,	$c = 85·785$,	$B = 39° 49'$.
25.	$a = 147·4$,	$c = 68·5$,	$C = 27° 42'$.
26.	$a = 2403$,	$b = 2521·1$,	$A = 72° 24'$.
27.	$b = 26·64$,	$c = 31·57$,	$B = 57° 47'$.
28.	$b = 599·6$,	$c = 250·5$,	$C = 24° 42'$.

29.	$a = 110{\cdot}9$,	$b = 22{\cdot}73$,	$B = 11°\ 50'$.
30.	$a = 165{\cdot}1$,	$b = 134{\cdot}07$,	$C = 35°\ 42'$.
31.	$a = 75{\cdot}41$,	$b = 16{\cdot}02$,	$C = 77°\ 44'$.
32.	$a = 31{\cdot}77$,	$b = 33{\cdot}77$,	$C = 19°\ 50'$.
33.	$b = 64{\cdot}41$,	$c = 16{\cdot}9$,	$A = 74°\ 45'$.
34.	$b = 508{\cdot}84$,	$c = 354{\cdot}52$,	$A = 45°\ 50'$.
35.	$a = 39{\cdot}441$,	$c = 63{\cdot}684$,	$B = 51°\ 44'$.
36.	$a = 536{\cdot}19$,	$b = 799{\cdot}92$,	$A = 33°\ 50'$.
37.	$a = 973{\cdot}5$,	$b = 229{\cdot}2$,	$A = 76°\ 47'$.
38.	$b = 4{\cdot}982$,	$c = 26{\cdot}536$,	$C = 79°\ 22'$.
39.	$b = 353{\cdot}54 = c$,		$C = 45°$.
40.	$a = 720$,	$c = 623{\cdot}54$,	$B = 30°$.
41.	$b = 808{\cdot}3$,	$c = 404{\cdot}15$,	$A = 60°$.
42.	$b = 1989{\cdot}7$,	$c = 1922{\cdot}1$,	$A = 15°$.
43.	$b = 1585{\cdot}4$,	$c = 1436{\cdot}8$,	$A = 25°$.

Examples XVIII. (Page 237.)

3. $-\frac{1}{16}$. **6.** Less. **14.** 10 and ∞. **16.** $\frac{1}{7}\sqrt{7}$.

17. $\frac{1}{50}\sqrt{230}$, $4\sqrt{2}/\sqrt{115}$. **18.** $\frac{9}{7}$. **23.** $\frac{3}{5}$, $\frac{4}{5}$, 1.

24. $\frac{4}{5}$, $\frac{56}{65}$, $\frac{12}{13}$. **25.** $\frac{40}{41}$, $\frac{24}{25}$, $\frac{496}{1025}$. **26.** 1470, 1140, 216, 84, 630.

31. 1000 ft.; $PA = 500\,(\sqrt{6}-\sqrt{2})$ ft., $PB = 1000$ ft. $= QB$.
$QA = 1000\sqrt{2}$ ft.

34. 270 sq. ft.; $\sin A = \frac{5}{13} = \cos B$, etc. **40.** 120°; 60, 100, 140 ft.

43. $\sin^2\theta.\tan\frac{1}{2}A = -d\pm\sqrt{(d^2+\cos^2\theta.\sin^2\theta)}$; where $\theta = \frac{1}{2}(B-C)$,
$d = p/a$. **56.** 5314·1 sq. ft. **57.** $-\frac{1}{16}$; 93° 35′.

58. 111640 sq. ft. **59.** 37° 22′. **60.** 195880 sq. ft.

61. $\frac{1}{6}\sqrt{6}$; 48° 12′. **62.** −0·65; 130° 32′. **63.** 73° 24′.

Examples XIX. (Page 251.)

1. 4044·3 ft. **2.** 255·38. **3.** 1364·6. **4.** 313·47.

5. 36° 8′. **6.** 74° 19′, 51° 59′. **7.** 72° 37′, 56° 3′.

8. 88° 31′, 33° 31′. **9.** 73° 2′, 48° 42′.

10. 68° 25′, 37° 15′. **12.** 226·87. **13.** 25·3.

14. 39° 23′, 12° 57′. **15.** 65° 30′, 30° 6′.

16. 74·58. **17.** 70° 54′, 49° 6′. **18.** 93° 35′.

19. 48° 11′. **20.** 78° 28′. **21.** 41° 17′.

22. 71° 46′, 46° 26′, 61° 48′.

23. 62° 31′ and 102° 18′, or 117° 29′ and 47° 20′.

24. 5926·6. **25.** 3003. **26.** 80·4.

27. 16° 6′; 317790 sq. ft.

29. (i) $A = 75° 27'$, $B = 41° 49'$, $C = 62° 44'$.

(ii) $A = 20° 55'$, $B = 41° 49'$, $C = 117° 16'$.

30. $b = 767·8$, $c = 1263·6$, $A = 106° 15'$.

31. $b = 12413$, $c = 9021$, $C = 36° 18'$.

32. $B = 71° 44'$, $C = 48° 16'$, $a = 12·77$.

33. $A = 33° 49'$, $B = 109° 11'$, $c = 307$.

34. $A = 19° 11'$, $B = 61° 13'$, $C = 99° 36'$.

35. $B = 50° 38'$, $C = 51° 47'$, $c = 6318$.

36. $a = 279·8$, $b = 243$, $C = 91° 43'$.

37. 32108. **38.** 172·64. **39.** 148° 8′, 6° 22′.

40. 123° 43′, 12° 17′. **41.** 108° 36′, 31° 24′. **42.** 132° 35′.

43. 91° 5′. **44.** 55° 46′. **45.** 63° 31′. **46.** 32° 26′, 106° 24′.

47. 49° 16′, 10° 44′. **48.** 96° 27′ or 19° 3′.

49. $B = 65° 59'$, $C = 41° 56'$.

50. $C = 96° 27'$, $a = 595·5$, $b = 739·2$.

51. $C = 142° 44'$, $a = 151·3$, $b = 144·2$.

52. $A = 56° 48'$, $a = 89·8$, $b = 95·7$.

53. $C = 62°$, $a = 1877$, $b = 589$.

54. $C = 151°$, $a = 17·96$, $b = 19·22$.

55. Triangle impossible.

56. $B = 16° 41'$ or $163 19'$, $C = 148° 5'$ or $1° 27'$,
$c = 368·3$ or $17·7$.

57. $B = 18° 38'$ or $161° 22'$, $C = 142° 46'$ or $2'$,
$c = 3596$ or $3·08$.

58. $A = 2° 8'$, $B = 160° 36'$, $b = 107·4$.

59. $A = 4° 58'$, $C = 150° 2'$, $c = 98·1$.

60. $B = 116° 4'$, $C = 48° 28'$, $a = 5·34$.

61. $B = 109° 43'$, $C = 42° 40'$, $a = 12·3$.

62. $B = 123° 29'$, $C = 19° 29'$, $a = 64·99$.

63. $A = 13° 33'$, $C = 144° 17'$, $c = 44·86$.

64. $A = 83° 54'$, $B = 83° 52'$, $c = 2635$.

65. $A = 18° 12'$, $B = 33° 7'$, $C = 128° 41'$.

66. $A = 46° 34'$, $B = 57° 55'$, $C = 75° 31'$.
67. $A = 49° 28'$, $B = 58° 45'$. $C = 71° 47'$.
68. $A = 47° 49'$, $B = 53° 12'$, $C = 78° 59'$.
69. $A = 7° 16'$, $B = 164° 50'$, $C = 7° 54'$.

Examples XX. (Page 260.)

2. $\sqrt{3}$, $\frac{1}{2}(3-\sqrt{3})$, $3-\sqrt{3}$ ml.

3. $\mathbf{PA} = bc \sin(A-\alpha)/\sqrt{(b^2 \sin^2 \overline{\alpha+\beta}+c^2 \sin^2 \beta}$
$\qquad -2bc \sin \beta . \sin \overline{\alpha+\beta} . \cos \overline{A-\alpha})$;
$\mathbf{PB} = ca \sin(B-\beta)/\sqrt{(c^2 \sin^2 \alpha+a^2 \sin^2 \overline{\alpha+\beta}}$
$\qquad -2\,ca \sin \alpha . \sin \overline{\alpha+\beta} . \cos \overline{B-\beta})$;
$\mathbf{PC} = ab \sin(\alpha+\beta+C)/\sqrt{(a^2 \sin^2 \beta+b^2 \sin^2 \alpha}$
$\qquad +2ab \sin \alpha . \sin \beta . \cos \overline{\alpha+\beta+C})$;
where $\alpha = \angle \mathbf{BPC}$, $\beta = \angle \mathbf{APC}$.

4. 120 ft. **5.** $2/\sqrt{(4-\sqrt{3})}$ ml.; $\frac{1}{13}(7+5\sqrt{3})$ ml.

6. $\sin^{-1} \frac{5}{13}$ E. of N.

7. $\tan^{-1}(\tan a . \cos b)$; $L \tan \theta = L \tan a + L \cos b - 10$.

9. $100(3-\sqrt{3})/\sqrt{2}$ yd.

10. $\cos \mathbf{POA} = \sin \theta \sqrt{(a^2+r^2 \sin^2 \theta)}/r - \cos^2 \theta$; $\sin \theta < r/\sqrt{(a^2+2r^2)}$.

11. $15(\sqrt{401}-1)$ ft. **12.** $h \cot \phi . \tan \theta$. **14.** 2·458 ml.

15. 1070·6 ft.; 1758·5 ft. **16.** 722·85 ft.; 29° 5′.

17. 1960·9 yd. **18.** 1701·3 ft. **19.** 37·72 ft.

20. 134·77 ft. **21.** 56° 19′, 33° 41′.

22. 1606·5 ft. **24.** 1696 ft. **25.** 140·98.

26. 64·09 yd. **27.** 29·72 ft. **28.** 1804·1 ft.

Examples XXI. (Page 273.)

11. (i) a, $\frac{5}{2}a$; (ii) $\frac{1}{2}\sqrt{7}$, $\frac{8}{7}\sqrt{7}$. **12.** 4, $10\frac{1}{2}$, 12, 14. **13.** Equal.

14. 12, 6, 28. **15.** 12, 16, 20. **16.** 2 : 1.

Examples XXII. (Page 286.)

1. 4 and 6. **2.** 8 ft. **3.** 3·192 ft.

5. (i) $\frac{45}{4}\sqrt{(1+\frac{2}{5}\sqrt{5})}$; (ii) $150\sqrt{3}$; (iii) $\frac{567}{4} \cot \frac{1}{7}\pi$;
(iv) $128(\sqrt{2}+1)$; (v) $324 \cot \frac{1}{9}\pi$ (sq. ft. in each case).

6. $2 : \sqrt{5}$. **7.** $(p^2/4n) \cot(\pi/n)$. **8.** $4-2\sqrt{2} : 1$.

10. 1008 sq. ft. **19.** 111640 sq. ft.

20. $\frac{1}{2}a^2 \sin \theta$; 195880 sq. ft. **21.** 34° 43′; 5·45 ft.

Examples XXIII. (Page 297.)

1. $1257\frac{1}{7}$ sq. ft.; $31\frac{3}{7}$ sq. ft. **2.** 21·8743 sq. ft. **3.** $28\frac{2}{7}$ sq. in.
4. 1386 sq. cm. **5.** $229\frac{1}{11}°$. **6.** 15·36 sq. in.
7. $4\pi-3\sqrt{3} : 2\pi+3\sqrt{3}$. **8.** $2r^2(3\sqrt{3}-\pi)$.
9. Equal, as far as the data go. **10.** $p^2/4\pi$, $p^2/12\sqrt{3}$.
13. $(\sqrt{3}-\frac{1}{2}\pi)$ sq. ft. **14.** $(4\sqrt{3}-\frac{11}{6}\pi)\,r^2$.
15. 17·5 in. **16.** 28·4 in. **18.** $\cos A - x \sin A$.
21. (i) ·60355, ·65328; (ii) ·62842, ·64073; (iii) ·63458, ·63764; (iv) ·63649, ·63668. **23.** ·000097.
24. 200 ft. **25.** 95,260,000 ml. **26.** 880,000 ml.
27. 238,600 ml. **28.** 50 billion ml. **29.** $17\frac{2}{11}''$.
31. 110:1. **34.** 1 ml. **35.** $2{\cdot}6'$. **36.** 140·7 ml.
37. 4,000 ml. **38.** 22 ml. **39.** 6·71 ml.; 6·71 ml.
41. 29·709. **42.** 38·52 ml. **43.** $\theta+\frac{1}{3}\theta^3$.

PRINTED IN GREAT BRITAIN BY UNIVERSITY TUTORIAL PRESS LTD, FOXTON NEAR CAMBRIDGE